

# On the Dirichlet problem for asymmetric zero range process on increasing domains.

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## Abstract

We characterize the principal eigenvalue of the generator of the asymmetric zero-range process on  $\mathbb{Z}^d$  in dimensions  $d \geq 3$ , with Dirichlet boundary on special domains. We obtain a Donsker-Varadhan variational representation for the principal eigenvalue, and show that the corresponding eigenfunction is unique in a natural class of functions. This allows us to obtain asymptotic hitting time estimates.

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*Running head:* Principal eigenvalue for asymmetric zero range.

## 1 Introduction

We are concerned in this work with obtaining hitting time estimates for the asymmetric zero-range process (AZRP). For this purpose, we consider the problem of characterizing the principal eigenvalue and principal eigenfunctions of the generator of AZRP, denoted by  $\mathcal{L}$ , with Dirichlet boundary on special domains. Though  $\mathcal{L}$  is neither compact, irreducible, nor self-adjoint, its physical origin endows crucial monotonicity properties.

The AZRP models the conservative evolution of charged particles interacting over short range, in an electrical field. Thus, this process denoted by  $\{\eta_t, t \geq 0\}$ , lives on  $\{\eta : \eta(i) \in \mathbb{N}, i \in \mathbb{Z}^d\}$ , and evolves informally as follows. At time zero and at each site  $i \in \mathbb{Z}^d$ , we draw a number of particles  $\eta(i) \in \mathbb{N}$ . To each particle we attach the trajectory of an asymmetric random walk with transition kernel  $\{p(i, j); i, j \in \mathbb{Z}^d\}$ . Now, each site  $i \in \mathbb{Z}^d$  has an independent exponential process, its *clock*, of intensity  $g(\eta_t(i))$  at time  $t$ , where  $g : \mathbb{N} \rightarrow [0, \infty)$  is increasing. When the clock of site  $i$  rings, say at time  $t$ , we choose a particle uniformly among the  $\eta_t(i)$  ones and we move it to its next position along its attached trajectory. The conservation of the particles number imposes a one-parameter

family of ergodic time-invariant measures  $\{\nu_\rho, \rho > 0\}$ , which happens to consist of product measures [1, 13]. The name zero-range is justified since only particles at the same site can interact with each other. Note also that  $g(k) = k$  corresponds to independent random walks with clocks' intensity 1.

A question motivated by physics is the time of occurrence of spots with large densities of particles, say  $\tau$ , when the gas is initially prepared with a homogeneous density. Thus, we consider a stationary process with respect to  $\nu_\rho$ , and focus on occurrence time of patterns of the type

$$\mathcal{A} := \{\eta : \sum_{i \in \mathcal{S}} \eta(i) > L\}, \quad (\text{and } \tau := \inf\{t : \eta_t \in \mathcal{A}\}) \quad (1.1)$$

where the support of  $\mathcal{A}$ ,  $\mathcal{S}$ , is a finite subset of  $\mathbb{Z}^d$ , and  $L$  a given integer.

The key feature of our model is that the partial order  $-\eta \prec \zeta$  meaning  $\eta(i) \leq \zeta(i)$  for all  $i \in \mathbb{Z}^d$  is preserved under the evolution. Another important feature is that the invariant measures  $\{\nu_\rho, \rho > 0\}$  all satisfy FKG's inequality, i.e. for  $f$  and  $g$  increasing functions

$$\int f g d\nu_\rho \geq \int f d\nu_\rho \int g d\nu_\rho. \quad (1.2)$$

This was the setting of [2] whose relevant results we now recall. A simple subadditive argument yielded the asymptotic rough estimate

$$\lambda(\rho) = - \lim_{t \rightarrow \infty} \frac{1}{t} \log(P_{\nu_\rho}(\tau > t)). \quad (1.3)$$

When the drift is nonzero,  $\lambda(\rho)$  is positive in any dimensions. Furthermore, if we denote by  $\mathcal{L}^*$  the dual of  $\mathcal{L}$  in  $L^2(\nu_\rho)$ , which corresponds to an AZRP with reversed drift, then when dimension  $d \geq 3$ , there exist  $u, u^* \in L^p(\nu_\rho)$  for any  $p \geq 1$  in the domain of  $\mathcal{L}$  and  $\mathcal{L}^*$  respectively, with

$$(i) \quad 1_{\mathcal{A}^c} \mathcal{L}(u) + \lambda(\rho)u = 0, \quad \text{and} \quad (ii) \quad 1_{\mathcal{A}^c} \mathcal{L}^*(u^*) + \lambda(\rho)u^* = 0, \quad (1.4)$$

However, and this was most unfortunate from a physical point of view, a link with finite dimensional dynamics was missing, as well as a variational representation for  $\lambda(\rho)$ . This is what we establish in this paper. Moreover, we establish uniqueness for  $u$  in some class of functions, which in turn yields an asymptotic estimate for the hitting time.

We have chosen to introduce some symbols intuitively so as to be able to state our main results postponing definitions and notations as much as possible to Section 2.

A way of defining the AZRP with initial law  $\nu_\rho$  on  $\mathbb{Z}^d$  is through a limit of irreducible processes, where particles evolve on  $[-n, n]^d$  as a zero-range process with creation and annihilation at the boundary. Informally, if  $\mathcal{F}_n$  is the  $\sigma$ -field generated by  $\{\eta(i), i \in [-n, n]^d\}$ , then we define

$$\mathcal{L}_n^\rho(\varphi) = E_{\nu_\rho}[\mathcal{L}(\varphi) | \mathcal{F}_n].$$

The generator  $\mathcal{L}_n^\rho$  will be shown to inherit the same property of monotonicity as  $\mathcal{L}$  and to have  $\nu_\rho$  as invariant measure. Thus, its principal Dirichlet eigenvalue  $\lambda_n(\rho)$  is obtained as in (1.3). We show in Section 3.4 that  $\mathcal{L}_n^\rho$  has a unique normalized eigenfunction  $u_n \geq 0$ , associated with  $\lambda_n(\rho)$ . Then, our main observation in Section 3.5 is the following.

**Lemma 1.1** For  $\lambda(\rho)$  given by (1.3), and  $\lambda_n(\rho)$  corresponding to  $\mathcal{L}_n^\rho$ , we have

$$\lim_{n \rightarrow \infty} \lambda_n(\rho) = \inf_n \lambda_n(\rho) = \lambda(\rho). \quad (1.5)$$

Moreover, we establish a link between finite and infinite volume eigenfunctions.

**Theorem 1.2** When  $d \geq 3$ ,  $\{u_n, n \in \mathbb{N}\}$  converges to a solution of (1.4(i)) in weak- $L^2(\nu_\rho)$ .

In [2], a solution of (1.4(i)) was obtained through another sequence, say  $\{u_t, t \geq 0\}$  in which  $u_t$  was the density (w.r.t  $\nu_\rho$ ) of the law of time-reversed process  $\eta_t^*$  conditioned on  $\{\tau > t\}$ . The functions  $\{u_t, t \geq 0\}$  were positive and decreasing on  $\mathcal{A}^c$ , and satisfied the following uniform bound: for site  $i$  large enough, if  $\epsilon_i$  is the probability that a random walk starting on  $i$  with transition kernel  $\{p(\cdot, \cdot)\}$  hits the support of  $\mathcal{A}$ , then when  $d \geq 3$

$$0 \leq u_t(\eta) - u_t(A_i^+ \eta) \leq \epsilon_i u_t(\eta), \quad (1.6)$$

where  $A_i^+$  adds a particle at  $i \in \mathbb{Z}^d$ .

We denote by  $\mathcal{D}_\rho$  the convex set of non-negative decreasing functions of finite integral (w.r.t  $\nu_\rho$ ), satisfying (1.6). We denote by  $\mathcal{D}_\rho^+$  the positive functions of  $\mathcal{D}_\rho$ . Finally, we define a dual space of probability measures,  $\mathcal{M}_\rho$ , absolutely continuous with respect to  $\nu_\rho$ , and whose density satisfies a condition similar to (1.6).

Intuitively, a Donsker-Varadhan's type functional would read  $\Gamma_\infty(\varphi, \mu) = \int \mathcal{L}(\varphi) / \varphi d\mu$  for  $(\varphi, \mu) \in \mathcal{D}_\rho^+ \times \mathcal{M}_\rho$ . One problem is that  $\mathcal{L}$  cannot be defined on  $\mathcal{D}_\rho^+$  as a convergent series. Thus, we define  $\Gamma_\infty(\varphi, \mu)$  in Proposition 4.3 of Section 4 as a Cauchy limit using cancelation due to gradient bounds (1.6) on  $\varphi$  and  $d\mu/d\nu_\rho$ .

We obtain in Section 4.2 a Donsker-Varadhan variational formula for the principal eigenvalue.

**Theorem 1.3** When  $d \geq 3$ , and  $\mathcal{A}$  is increasing with bounded support, we have

$$\lambda(\rho) = - \sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho^+} \Gamma_\infty(\varphi, \mu). \quad (1.7)$$

Obtaining (1.7) is linked with the issue of uniqueness of the principal eigenfunction, since the minimax theorem hidden behind Donsker-Varadhan formula requires a convex functional  $h \mapsto \Gamma_\infty(e^h, \mu)$ , on a convex set of functions regular enough. Note that  $\mathcal{D}_\rho^+$  is all the more appropriate since when written for  $h = \log(\varphi)$  with  $\varphi \in \mathcal{D}_\rho^+$ , condition (1.6) reads

$$h(\eta) \geq h(A_i^+ \eta) \geq h(\eta) + \log(1 - \epsilon_i) \quad (\text{when } \epsilon_i < 1), \quad (1.8)$$

and define a convex set. Now, the main uniqueness result is the following.

**Theorem 1.4** When  $d \geq 3$ , there is a unique normalized Dirichlet eigenfunction in  $\mathcal{D}_\rho$ . This eigenfunction is positive  $\nu_\rho$ -a.s. on  $\mathcal{A}^c$ .

The proofs of Theorem 1.4 and Theorem 1.2 are conducted in Section 5. We sketch the simple intuitive steps behind the proof of uniqueness. Assume there exist  $u, \tilde{u}$  solutions of (1.4(i)) in  $\mathcal{D}_\rho$ . Then, they are actually positive (on  $\mathcal{A}^c$ ), and satisfy

$$\forall \mu \in \mathcal{M}_\rho, \quad \Gamma_\infty(u, \mu) = \Gamma_\infty(\tilde{u}, \mu) = -\lambda(\rho). \quad (1.9)$$

As already mentionned, if  $u, \tilde{u} \in \mathcal{D}_\rho$  and  $\gamma \in ]0, 1[$ , then  $u_\gamma := u^\gamma \tilde{u}^{1-\gamma} \in \mathcal{D}_\rho$ . Now, by convexity of  $h \mapsto \Gamma_\infty(\exp(h), \mu)$

$$\forall \mu \in \mathcal{M}_\rho, \quad -\lambda(\rho) = \gamma \Gamma_\infty(u, \mu) + (1 - \gamma) \Gamma_\infty(\tilde{u}, \mu) \geq \Gamma_\infty(u_\gamma, \mu) \quad (1.10)$$

We now choose a special  $\mu$  so that equality obtains in (1.10). The space  $\mathcal{M}_\rho$  is built so that if  $u^*$  is a positive solution of (1.4(ii)), then

$$d\mu^* := \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho \in \mathcal{M}_\rho. \quad (1.11)$$

Now, by a formal use of duality

$$\Gamma_\infty(u_\gamma, \mu^*) = \left\langle \frac{\mathcal{L}(u_\gamma)}{u_\gamma}, \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho \right\rangle = \left\langle \frac{\mathcal{L}^*(u^*)}{u^*}, \frac{u_\gamma u^*}{\int u_\gamma u^* d\nu_\rho} d\nu_\rho \right\rangle = -\lambda(\rho). \quad (1.12)$$

Finally, the case of equality in (1.10) implies that  $\tilde{u}/u$  is  $\nu_\rho$ -a.s. constant on  $\mathcal{A}^c$  by using the triviality of the  $\sigma$ -field of exchangeable events under  $\nu_\rho$ .

As a consequence of Theorem 1.4, we obtain an asymptotic estimate of the first hitting time of  $\mathcal{A}$ . To link this last result with those of [2], we recall Corollary 2.8 of [2] which was based on  $L^p(\nu_\rho)$  estimates for  $u$  and  $u^*$ . When  $d \geq 3$ , there is a positive constant  $\underline{c}$  such that for any  $t \geq 0$ ,

$$\underline{c} \leq \exp(\lambda(\rho)t) P_{\nu_\rho}(\tau > t) \leq 1. \quad (1.13)$$

As a corollary of the uniqueness of the principal eigenfunction in  $\mathcal{D}_\rho$ , we obtain the following estimates whose proof makes up Section 6.

**Theorem 1.5** *When  $d \geq 3$ ,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{\lambda(\rho)s} P_{\nu_\rho}(\tau > s) ds = \frac{1}{\int u u^* d\nu_\rho}. \quad (1.14)$$

## 2 Notations and preliminaries.

We first recall in Section 2.1, the hypotheses needed to define the AZRP on  $\mathbb{Z}^d$ . Then, in Section 2.2, we describe the class of patterns we consider here. Section 2.3 contains the definition of all function spaces which we use.

## 2.1 The zero-range process

The transition kernel  $\{p(i, j), i, j \in \mathbb{Z}^d\}$  is associated with a single-particle trajectory and satisfies for all  $i, j$  in  $\mathbb{Z}^d$

- (i)  $p(i, j) \geq 0$ ,  $p(i, i) = 0$ ,  $\sum_{i \in \mathbb{Z}^d} p(0, i) = 1$ .
  - (ii)  $p(i, j) = p(0, j - i)$  (translation invariance).
  - (iii)  $p(i, j) = 0$  if  $|i - j| > R$  for some fixed  $R$  (finite range).
  - (iv) If  $p_s(i, j) = p(i, j) + p(j, i)$ , then  $\forall i \in \mathbb{Z}^d, \exists n, p_s^{(n)}(0, i) > 0$  (irreducibility).
  - (v)  $\sum_{i \in \mathbb{Z}^d} i p(0, i) \neq 0$  (positive drift).
- (2.1)

Note that by (i) and (ii), the transition kernel  $p(., .)$  is doubly stochastic. Thus, we can introduce a *dual* transition kernel  $\{p^*(i, j), i, j \in \mathbb{Z}^d\}$ , with  $p^*(i, j) = p(j, i)$ .

We also need a particle dependent intensity  $g$  which satisfies

- (i)  $g : \mathbb{N} \rightarrow [0, \infty)$  is increasing.
  - (ii)  $g(0) = 0, g(1) = 1$  (normalization).
  - (iii)  $\Delta := \sup_k (g(k+1) - g(k)) < \infty$ .
- (2.2)

For notational simplicity, we call the intensity at site  $i \in \mathbb{Z}^d$ ,  $g_i(\eta) := g(\eta(i))$ .

For any  $\gamma \in [0, \sup_k g(k)[$ , we define a probability  $\theta_\gamma$  on  $\mathbb{N}$ , by

$$\theta_\gamma(0) = 1/Z(\gamma), \quad \text{and when } n \neq 0, \quad \theta_\gamma(n) = \frac{1}{Z(\gamma)} \frac{\gamma^n}{g(1) \dots g(n)}, \quad (2.3)$$

where  $Z(\gamma)$  is the normalizing factor. If we set  $\rho(\gamma) := \sum_{n=1}^{\infty} n \theta_\gamma(n)$ , then  $\rho : [0, \sup_k g(k)[ \rightarrow [0, \infty[$  is increasing. Let  $\gamma(\cdot)$  be the inverse of  $\rho(\cdot)$ , and for a constant density  $\rho > 0$ , let  $\nu_\rho$  be the product probability with marginal law  $\theta_{\gamma(\rho)}$ . Thus, we have

$$\forall B \subset \mathbb{Z}^d, \quad \int \prod_{i \in B} \eta(i) d\nu_\rho = \rho^{|B|}, \quad \text{and} \quad \int g_i(\eta) \varphi(A_i^- \eta) d\nu_\rho(\eta) = \gamma(\rho) \int \varphi d\nu_\rho, \quad (2.4)$$

where  $A_i^- \eta$  has one particle less than  $\eta$  at site  $i$ . Also, we will often use that

$$0 \leq g(n) \leq \Delta n, \quad (\text{by (ii) and (iii) of (2.2)}), \quad \text{and} \quad \int g_i^p d\nu_\rho < \infty, \quad \text{for any } p \in \mathbb{N}. \quad (2.5)$$

Following [9], (see also [1] and [13] Section 2), let

$$\alpha(i) = \sum_{n=0}^{\infty} 2^{-n} p^n(i, 0), \quad \text{and for } \eta, \zeta \in \mathbb{N}^{\mathbb{Z}^d}, \quad \|\eta - \zeta\| = \sum_{i \in \mathbb{Z}^d} |\eta(i) - \zeta(i)| \alpha(i).$$

Since the transition kernel  $p$  is finite range (by 2.1(iii)), another possible choice is  $\alpha(k) = \exp(-(|k_1| + \dots + |k_d|))$  for any site  $k = (k_1, \dots, k_d)$  (see [9]). Our state space is  $\Omega = \{\eta : \|\eta\| < \infty\}$ , and we call  $\mathbb{L}$  the space of Lipschitz functions from  $(\Omega, \|\cdot\|)$  to  $(\mathbb{R}, |\cdot|)$ , and  $\mathbb{L}_b$  the subspace of  $\mathbb{L}$  consisting of bounded functions. For  $\varphi \in \mathbb{L}$ , we call

$$L(\varphi) := \sup \left\{ \frac{|\varphi(\eta) - \varphi(\xi)|}{\|\eta - \xi\|} : \|\eta - \xi\| > 0, \eta, \xi \in \Omega \right\}. \quad (2.6)$$

In [1], it is shown that a semi-group can be constructed on  $\mathbb{L}$  with formal generator

$$\mathcal{L}\varphi(\eta) := \sum_{i,j \in \mathbb{Z}^d} p(i,j)g(\eta(i)) (\varphi(T_j^i \eta) - \varphi(\eta)), \quad (2.7)$$

where  $T_j^i \eta(k) = \eta(k)$  if  $k \notin \{i,j\}$ ,  $T_j^i \eta(i) = \eta(i) - 1$ , and  $T_j^i \eta(j) = \eta(j) + 1$ . If we set  $\nabla_j^i \varphi = \varphi \circ T_j^i - \varphi$ , we will often use that on  $\{\eta(i) > 0\}$

$$\nabla_j^i \varphi = (\varphi \circ \mathbf{A}_j^+ - \varphi \circ \mathbf{A}_i^+) \circ \mathbf{A}_i^-. \quad (2.8)$$

Thus, if we set  $\Delta_j^i \varphi = \varphi \circ \mathbf{A}_j^+ - \varphi \circ \mathbf{A}_i^+$ , and use (2.5) and (2.8), we have the following integration by parts formula

$$\int g_i \nabla_j^i(\varphi) f d\nu_\rho = \gamma_\rho \int \Delta_j^i(\varphi) \mathbf{A}_i^+(f) d\nu_\rho. \quad (2.9)$$

Also, for convenience, we often write  $\mathbf{A}_i^\pm \varphi$  for  $\varphi \circ \mathbf{A}_i^\pm$ .

In [13] Section 2,  $\mathcal{L}$  is extended to a generator, again called  $\mathcal{L}$  for convenience, on  $L^2(\nu_\rho)$  for any  $\rho > 0$ . It is also shown that  $\mathbb{L}_b$  is a core for  $\mathcal{L}$ . Moreover,  $\{\nu_\rho, \rho > 0\}$  are ergodic invariant measures for  $\mathcal{L}$ . We denote by  $\mathcal{D}(\mathcal{L}, L^2(\nu_\rho))$  the domain of  $\mathcal{L}$  in  $L^2(\nu_\rho)$ , and by  $\|\cdot\|_\nu$  the  $L^2(\nu)$ -norm, for any probability measure  $\nu$ . Finally, we consider the adjoint (or time-reversed) of  $\mathcal{L}$  in  $L^2(\nu_\rho)$ , acting on Lipschitz functions  $\varphi$  and  $\psi$  by

$$\int \mathcal{L}^*(\varphi) \psi d\nu_\rho := \int \varphi \mathcal{L}(\psi) d\nu_\rho. \quad (2.10)$$

With our hypothesis,  $\mathcal{L}^*$  is again the generator of a zero-range process with transition kernel  $p^*(.,.)$  satisfying  $p^*(i,j) := p(j,i)$  and with the same function  $g$ . We denote by  $\{S_t^*\}$  the associated semi-group, and by  $P_\eta^*$  the associated Markov process with initial configuration  $\eta \in \Omega$ .

## 2.2 Special patterns.

We first recall that there is a partial order on  $\Omega$ . For  $\eta, \xi \in \Omega$ , we say that  $\eta \prec \xi$  if  $\eta(i) \leq \xi(i)$  for all  $i \in \mathbb{Z}^d$ . A function  $f : \Omega \rightarrow \mathbb{R}$  is increasing if for  $\eta \prec \xi$ ,  $f(\eta) \leq f(\xi)$ . Also, we say that  $A \subset \Omega$  is increasing if its indicator  $1_A$  is increasing. Finally, for given probability measures  $\nu, \mu$  on  $\Omega$ , we say that  $\nu \prec \mu$  if  $\int f d\nu \leq \int f d\mu$  for every increasing function  $f$ . The zero-range process is a monotone process, i.e. there is a coupling such that  $P_{\eta,\zeta}(\eta_t \leq \zeta_t, \forall t) = 1$  whenever  $\eta \leq \zeta$ .

We will be concerned with the hitting time of pattern,  $\mathcal{A}$ , with the following properties dubbed  $(\mathcal{C}-\mathcal{F})$  for connectedness and finiteness:

- (i) It is non-empty, and its support  $\mathcal{S}$  is bounded. Thus,  $\nu_\rho(\mathcal{A}) > 0$ .
- (ii) It is increasing, and  $0_{\mathcal{S}} := \{\eta : \eta(i) = 0, \forall i \in \mathcal{S}\} \notin \mathcal{A}$ . Thus,  $\nu_\rho(\mathcal{A}) < 1$ .

- (iii) Its complement,  $\mathcal{A}^c$ , is connected, and is partitioned into a finite number of cylinders with support in  $\mathcal{S}$ , whose set we denote by  $\Theta$ . In other words, for any cylinder  $\theta \in \Theta$ , there is an integer  $n$ , a sequence  $\theta_0, \dots, \theta_n \in \Theta$ , and  $i_1, \dots, i_n \in \mathcal{S}$  such that

$$\theta_0 := 0_{\mathcal{S}}, \quad \theta_n = \theta, \quad \text{and} \quad \theta_k = \mathbf{A}_{i_k}^+ \theta_{k-1} \quad \text{for } k = 1, \dots, n.$$

A typical example of patterns satisfying  $(\mathcal{C}-\mathcal{F})$  is given in (1.1). Note also that if  $\mathcal{A}$  satisfies  $(\mathcal{C}-\mathcal{F})$ , there is an integer  $L$  such that  $\{\eta : \sum_{\mathcal{S}} \eta(i) > L\} \subset \mathcal{A}$ .

We denote by  $\bar{\mathcal{L}} := 1_{\mathcal{A}^c} \mathcal{L}$  and  $\{\bar{S}_t, t \geq 0\}$ , respectively the generator and associated semi-group for the process killed on  $\mathcal{A}$ .

## 2.3 Function spaces.

The topology on  $\{\eta : \eta(i) \in \mathbb{N}, i \in \mathbb{Z}^d\}$ , is the product of discrete topology, so that  $\{\eta_n, n \in \mathbb{N}\}$  converges to  $\eta$ , if for any site  $i \in \mathbb{Z}^d$ , there is  $n_0$  such that for  $n \geq n_0$   $\eta_n(i) = \eta(i)$ .

Let  $H_{\mathcal{S}} := \inf\{t : X_t \in \mathcal{S}\}$  for  $\{X_t\}$  a random walk with transition kernel  $\{p(i, j); i, j \in \mathbb{Z}^d\}$ . Note that  $\epsilon_i := \mathbb{P}_i(H_{\mathcal{S}} < \infty) \rightarrow 0$  as  $\|i\| \rightarrow \infty$ , (as well as  $\epsilon_i^*$  corresponding to a reversed drift) and when the dimension  $d \geq 3$ , then we have the classical results

$$\sum_{i \in \mathbb{Z}^d} \epsilon_i^2 + (\epsilon_i^*)^2 < \infty.$$

Let  $\mathcal{A}$  satisfy  $(\mathcal{C}-\mathcal{F})$ . Choose  $n$  large enough so that  $\mathcal{S} \subset \Lambda_n := [-n, n]^d$ , and set  $\Omega_n = \{\eta : \Lambda_n \rightarrow \mathbb{N}\}$ , and  $\mathcal{F}_n := \sigma(\{\eta(i), i \in \Lambda_n\})$ . We often make the abuse of considering functions on  $\Omega_n$  as defined also on  $\Omega_m$  for  $m \geq n$ , but depending only on the sites of  $\Lambda_n$ .

### 2.3.1 Functions on $\Omega_n$ .

A function  $\varphi$  on  $\Lambda_n$  with  $\varphi|_{\mathcal{A}} \equiv 0$  belongs to  $\mathcal{D}_n$  when

$$\begin{aligned} (0) \quad & 0 \leq \varphi, \\ (i) \quad & \forall \eta, \zeta \in \Omega_n \setminus \mathcal{A}, \text{ if } \eta \prec \zeta \quad \text{then} \quad \varphi(\zeta) \leq \varphi(\eta), \\ (ii) \quad & \forall \eta \in \Omega_n \setminus \mathcal{A}, \forall i \in \Lambda_n \setminus \mathcal{S}, \quad \varphi(\eta) - \varphi(\mathbf{A}_i^+ \eta) \leq \varphi(\eta) \epsilon_i, \\ (iii) \quad & \int \varphi d\nu_{\rho} < \infty. \end{aligned} \tag{2.11}$$

When  $\epsilon_i^*$  replaces  $\epsilon_i$  in (ii), we say that  $\varphi$  belong to  $\mathcal{D}_n^*$ . Also, we set  $\mathcal{D}_n^+ := \mathcal{D}_n \cap \{\varphi \text{ positive on } \mathcal{A}^c\}$ .

**Lemma 2.1**  *$\mathcal{D}_n$  is a convex subset of  $\mathbb{L}_b$ . When  $d \geq 3$ , if  $\varphi \in \mathcal{D}_n^+$ , then  $\varphi$  and  $1_{\mathcal{A}^c}/\varphi$  are in  $L^p(\nu_{\rho})$  for any  $p \geq 1$ .*

*Proof.* If  $\varphi \in \mathcal{D}_n$ , note that  $\varphi$  is bounded since  $0 \leq \varphi(\eta) \leq \varphi(0_{\Lambda_n})$ , where  $0_{\Lambda_n}$  is the empty configuration of  $\Omega_n$ . Take  $\eta, \zeta \in \Omega_n \setminus \mathcal{A}$ , and let  $\xi = \eta \vee \zeta - \eta \wedge \zeta$ , and set  $m = \sum_i \xi(i)$ . Since  $\varphi$  is decreasing

$$|\varphi(\eta) - \varphi(\zeta)| \leq \varphi(\eta \wedge \zeta) - \varphi(\eta \vee \zeta).$$

Now, let  $\{\eta_i, i = 0, \dots, m\}$  be an ordered sequence with

$$\eta \wedge \zeta = \eta_0 \prec \eta_1 \prec \dots \prec \eta_m = \eta \vee \zeta, \quad \text{with} \quad \eta_i = \mathbf{A}_{j_i}^+ \eta_{i-1},$$

where  $\{j_i, i = 1, \dots, m\}$  are the positions of the  $m$  particles of  $\xi$ . Then,

$$\varphi(\eta \wedge \zeta) - \varphi(\eta \vee \zeta) \leq \sum_{i=0}^{n-1} \varphi(\eta_i) - \varphi(\eta_{i+1}) \leq \sum_{i=1}^n \varphi(\eta_{i-1}) \epsilon_{j_i}.$$

We use that  $\varphi(\eta_i) \leq \varphi(0_{\Lambda_n})$ , and that  $\sum_i \epsilon_{j_i} = \sum_k \epsilon_k \xi(k)$ . Thus,

$$|\varphi(\eta) - \varphi(\zeta)| \leq \varphi(0_{\Lambda_n}) \sum_{k \in \Lambda_n} \epsilon_k \xi(k) \leq \varphi(0_{\Lambda_n}) \sup_{k \in \Lambda_n} \left( \frac{\epsilon_k}{\alpha_k} \right) \sum_{k \in \Lambda_n} \xi(k) \alpha(k). \quad (2.12)$$

Now, if  $\eta, \zeta \in \mathcal{A}$ , then (2.12) holds. Assume that  $\eta \in \Omega_n \setminus \mathcal{A}$  but  $\zeta \in \mathcal{A}$ . Inequality (2.12) follows once we notice that  $\|\eta - \zeta\| \geq \inf_{\mathcal{S}} \alpha > 0$ . Thus,  $\varphi$  is a Lipschitz bounded function. Now,  $\varphi$  and  $1_{\mathcal{A}^c}/\varphi$  are in  $L^p(\nu_\rho)$  for any integer  $p$  by Lemmas 7.2 and 7.4 of the Appendix.  $\blacksquare$

For any  $\varphi \in \mathcal{D}_n^+$ , we can define its logarithm on  $\mathcal{A}^c$ ,  $h = \log(\varphi)$ ; on  $\mathcal{A}$  we set  $h \equiv -\infty$ . Note that (2.11) reads for  $h$

$$\begin{aligned} \text{(i)} \quad & \forall \eta, \zeta \in \Omega_n \setminus \mathcal{A}, \text{ if } \eta \prec \zeta \quad \text{then} \quad h(\zeta) \leq h(\eta), \\ \text{(ii)} \quad & \forall \eta \in \Omega_n \setminus \mathcal{A}, \forall i \in \Lambda_n \setminus \mathcal{S}, \quad h(\mathbf{A}_i^+ \eta) \geq h(\eta) + \log(1 - \epsilon_i), \\ \text{(iii)} \quad & \int \exp(h) d\nu_\rho < \infty. \end{aligned} \quad (2.13)$$

Thus, we will say that  $h \in \mathcal{E}_n$  if it satisfies (2.13). A key and simple observation is the following.

**Lemma 2.2**  $\mathcal{E}_n$  is a convex set.

*Proof.* Inequalities (2.13) (i) and (ii) are stable under convex combination. Also, for  $\gamma \in ]0, 1[$ , and  $h_1, h_2 \in \mathcal{E}_n$  by Hölder inequality

$$\int \exp(\gamma h_1 + (1 - \gamma) h_2) d\nu_\rho \leq \left( \int e^{h_1} d\nu_\rho \right)^\gamma \left( \int e^{h_2} d\nu_\rho \right)^{1-\gamma} < \infty. \quad (2.14)$$

$\blacksquare$

We now define  $\mathcal{M}_n$  a space of probability measures whose elements have a density with respect to  $\nu_\rho$ , generically noted  $f$  satisfying: (i)  $f$  is decreasing on  $\mathcal{A}^c$ ,  $f|_{\mathcal{A}} \equiv 0$ , and

$$\text{(ii)} \quad \forall \eta \in \Omega_n \setminus \mathcal{A}, \forall i \notin \mathcal{S} \quad f(\eta) - f(\mathbf{A}_i^+ \eta) \leq f(\eta)(\epsilon_i + \epsilon_i^*) \quad (2.15)$$

**Lemma 2.3** Assume that  $d \geq 3$ .  $\mathcal{M}_n$  is a convex and compact set in the weak topology.



*Proof.* The convexity of  $\mathcal{M}_n$  is obvious. Consider the compact decreasing set

$$K_M = \{\eta \in \Omega_n : \eta(i) \leq M, \forall i \in \Lambda_n\}. \quad (2.16)$$

Note that  $\mathcal{M}_n$  is tight:

$$\lim_{M \rightarrow \infty} \sup_{\mu \in \mathcal{M}_n} \mu(K_M^c) = 0.$$

Indeed, since  $d\mu/d\nu_\rho$  is decreasing for any  $\mu \in \mathcal{M}_n$ , by FKG's inequality

$$\forall \mu \in \mathcal{M}_n, \quad \mu(K_M^c) = \int \mathbf{1}_{K_M^c} \frac{d\mu}{d\nu_\rho} d\nu_\rho \leq \nu_\rho(K_M^c) \xrightarrow{M \rightarrow \infty} 0.$$

Let  $\{\mu_n, n \in \mathbb{N}\}$  be in  $\mathcal{M}_n$ , with densities  $\{f_n := d\mu_n/d\nu_\rho\}$ . Let  $\{\mu_{n_k}\}$  a converging subsequence to  $\mu$ . For any  $\eta \in \Omega_n$ ,  $\mathbf{1}_\eta$  is a bounded continuous function, so that

$$f_{n_k}(\eta)\nu_\rho(\eta) = \int \mathbf{1}_\eta d\mu_{n_k} \xrightarrow{k \rightarrow \infty} \mu(\eta) = f(\eta)\nu_\rho(\eta). \quad (2.17)$$

Thus,  $f_{n_k}$  converges pointwise to  $f$  on  $\Omega_n$ . It is clear that  $f$  satisfies (2.15) so that  $\mu \in \mathcal{M}_n$ . ■

An important feature of  $\mathcal{M}_n$  is the following.

**Lemma 2.4** *Assume that  $d \geq 3$ . If  $\varphi \in \mathcal{D}_n^+$  and  $\varphi^* \in (\mathcal{D}_n^*)^+$ , then*

$$d\mu = \frac{\varphi\varphi^*d\nu_\rho}{\int \varphi\varphi^*d\nu_\rho} \in \mathcal{M}_n. \quad (2.18)$$

*Proof.* First, by Lemma 7.2,  $\int \varphi\varphi^*d\nu_\rho < \infty$ . Also, note that  $\varphi, \varphi^* > 0$  on  $\mathcal{A}^c$  so that  $\int \varphi\varphi^*d\nu_\rho > 0$ . Thus,  $\mu$  given in (2.18) is well defined. Now, since  $\varphi$  and  $\varphi^*$  are decreasing on  $\mathcal{A}^c$  and positive,  $d\mu/d\nu_\rho$  is decreasing on  $\mathcal{A}^c$ . Now, if  $\zeta = \mathbf{A}_i^+\eta$ , for  $i \notin \mathcal{S}$

$$\begin{aligned} \varphi(\eta)\varphi^*(\eta) - \varphi(\zeta)\varphi^*(\zeta) &= \varphi^*(\eta)(\varphi(\eta) - \varphi(\zeta)) + \varphi(\zeta)(\varphi^*(\eta) - \varphi^*(\zeta)) \\ &\leq \varphi(\eta)\varphi^*(\eta)(\epsilon_i + \epsilon_i^*). \end{aligned} \quad (2.19)$$

Thus,  $\mu$  satisfies (i) and (ii) of (2.15). ■

### 2.3.2 Functions on $\Omega$ .

We define  $\mathcal{D}_\rho$  as the natural extension of  $\mathcal{D}_n$  to functions defined on the whole of  $\Omega$ . Thus, functions in  $\mathcal{D}_\rho$  satisfy the inequalities in (2.11(0)-(iii)) but almost surely with respect to  $\nu_\rho$ . Also,  $\mathcal{D}_\rho^+$  denotes the functions of  $\mathcal{D}_\rho$  positive  $\nu_\rho$ -a.s. on  $\mathcal{A}^c$ . Similarly, we extend  $\mathcal{M}_n$  into  $\mathcal{M}_\rho$ , the space of probability measures absolutely continuous with respect to  $\nu_\rho$ , whose densities satisfy  $\nu_\rho$ -a.s. the same conditions as function of  $\mathcal{M}_n$ , but extended on the whole of  $\mathbb{Z}^d$ . Note that by linearity of the conditional expectation, for  $\varphi \in \mathcal{D}_\rho$ ,  $E_{\nu_\rho}[\varphi|\mathcal{F}_n] \in \mathcal{D}_n$ , and similarly if  $\mu \in \mathcal{M}_\rho$  with density  $f$ , then  $E_{\nu_\rho}[f|\mathcal{F}_n]d\nu_\rho \in \mathcal{M}_n$ .

**Lemma 2.5**  *$\mathcal{M}_\rho$  is compact in the weak topology.*

*Proof.* First, by Remark 7.3 of the Appendix, there is a constant  $C(\rho, 2) > 0$  such that

$$\sup_{\mu \in \mathcal{M}_\rho} \int \left( \frac{d\mu}{d\nu_\rho} \right)^2 d\nu_\rho \leq C(\rho, 2).$$

Recall that by Banach-Alaoglu Theorem,  $\{d\mu/d\nu_\rho, \mu \in \mathcal{M}_\rho\}$  is weak- $L^2(\nu_\rho)$  compact in  $L^2(\nu_\rho)$ . Secondly, for any  $\mu \in \mathcal{M}_\rho$  and integer  $n$ , as already mentionned

$$d\mu^{(n)} := E_{\nu_\rho} \left[ \frac{d\mu}{d\nu_\rho} \middle| \mathcal{F}_n \right] d\nu_\rho \in \mathcal{M}_n.$$

Now, let  $\{\mu_k, k \in \mathbb{N}\}$  be in  $\mathcal{M}_\rho$ , and let  $\mu_\infty$  be a weak- $L^2(\nu_\rho)$  limit along a subsequence, say  $\{n_k\}$ . Note that for each integer  $n$ , the following convergence holds in weak- $L^2(\nu_\rho)$

$$f_{n_k}^{(n)} := E_{\nu_\rho} \left[ \frac{d\mu_{n_k}}{d\nu_\rho} \middle| \mathcal{F}_n \right] \xrightarrow{k \rightarrow \infty} f_\infty^{(n)} := E_{\nu_\rho} \left[ \frac{d\mu_\infty}{d\nu_\rho} \middle| \mathcal{F}_n \right].$$

Moreover,  $f_\infty^{(n)} d\nu_\rho \in \mathcal{M}_n$ , since  $\mathcal{M}_n$  is compact by Lemma 2.3. Finally, the sequence  $\{f_\infty^{(n)}, n \in \mathbb{N}\}$  is a positive martingale which, by the martingale convergence Theorem, converges  $\nu_\rho$ -a.s. to  $f_\infty$ . Clearly, inequality (2.15) holds  $\nu_\rho$ -a.s. for  $f_\infty$ .  $\blacksquare$

**Remark 2.6** With the same arguments, we obtain that  $\mathcal{D}_\rho \cap \{\varphi : \int \varphi d\nu_\rho \leq c\}$  is weak- $L^2(\nu_\rho)$  compact, for any constant  $c > 0$ .

**Remark 2.7** We give now more details on how a solution  $u$  to (1.4(i)) was obtained in [2], and why  $u \in \mathcal{D}_\rho$  actually. We recall that for any probability  $\mu$ ,  $\Phi(\mu)$  introduced in [8] was the invariant measure of the renewal process corresponding to  $\{\eta_t\}$  started afresh from measure  $\mu$  each time it hits  $\mathcal{A}$ . Also, for any integer  $k$ , the map  $\Phi^{(k)}$  was the  $k$ -th iterates of  $\Phi$ . It is shown in Theorem 2.4 of [2] that the Cesaro weak- $L^2(\nu_\rho)$  limits of  $\{\Phi^{(k)}(\nu_\rho), k \in \mathbb{N}\}$  are solutions of (1.4(i)). There is actually a simple expression for  $\Phi^{(k)}$ . Since  $\lambda(\rho) > 0$ , we have  $\int_0^\infty P_{\nu_\rho}(\tau > t) t^k dt < \infty$ , and the following probability  $dm_k(t)$  on  $\{t \geq 0\}$  is well defined

$$dm_k(t) = \frac{P_{\nu_\rho}(\tau > t) t^k dt}{\int_0^\infty P_{\nu_\rho}(\tau > t) t^k dt} \quad \text{and} \quad \frac{d\Phi^{(k)}(\nu_\rho)}{d\nu_\rho}(\eta) = \int_0^\infty u_t(\eta) dm_k(t), \quad (2.20)$$

where  $u_t$  is mentionned in the paragraph preceding (1.6). Since,  $u_t \in \mathcal{D}_\rho$ , it is clear that for any integer  $k$ ,  $d\Phi^{(k)}(\nu_\rho)/d\nu_\rho \in \mathcal{D}_\rho$  as well as the Cesaro mean since  $\mathcal{D}_\rho$  is convex. Now, since  $\Phi^{(k)}(\nu_\rho)$  are probability measures, Remark 2.6 implies that all the Cesaro limits are in  $\mathcal{D}_\rho$ . Thus, there exists a solution of (1.4(i)) in  $\mathcal{D}_\rho$ : we denote it by  $u$ . Notice also that our uniqueness result, Theorem 1.4, implies that the whole Cesaro limit converges to  $u$ , thus strengthening the results of [2].

### 3 From finite domains to $\mathbb{Z}^d$ .

#### 3.1 Irreducible dynamics on $\Lambda_n$ .

Following the approach of [11], as in [1], we first consider, for any integer  $k$  and  $m$ , a finite-state generator  $\mathcal{L}_{(m)}^k$  on the hyper-surface

$$\Omega_{(m)}^k := \{\eta \in \mathbb{N}^{\Lambda_m} : \sum_{i \in \Lambda_m} \eta(i) = k\}.$$

For this purpose we introduce, for any integer  $n$  and for  $i, j \in \Lambda_n$

$$p_n(i, j) := \begin{cases} p(i, j) & \text{if } i \neq j \\ \sum_{k \notin \Lambda_n} p(i, k) & \text{if } i = j \end{cases}, \quad \text{and} \quad p_n^*(i, j) := \begin{cases} p^*(i, j) & \text{if } i \neq j \\ \sum_{k \notin \Lambda_n} p^*(i, k) & \text{if } i = j. \end{cases} \quad (3.1)$$

Note that  $\{p_n(i, j)\}$  is not doubly stochastic. The  $\{\mathcal{L}_{(m)}^k, k \in \mathbb{N}\}$  have the same expression, though on different domains

$$\forall \eta \in \Omega_{(m)}^k, \quad \mathcal{L}_{(m)}^k(\varphi)(\eta) = \sum_{i, j \in \Lambda_m} p_m(i, j) g_i(\eta) (\varphi(T_j^i \eta) - \varphi(\eta)). \quad (3.2)$$

The process generated by  $\mathcal{L}_{(m)}^k$  is well defined. Now, we take  $n < m - R$ , where  $R$  is the range of the transition kernel  $p(\cdot, \cdot)$ , and for  $\varphi \in \mathcal{D}_n$ , we define

$$\mathcal{L}_n^\rho(\varphi) = \lim_{K \rightarrow \infty} \sum_{k=0}^K E_{\nu_\rho} [1_{\Omega_{(m)}^k} \mathcal{L}_{(m)}^k(\varphi) | \mathcal{F}_{\Lambda_n}]. \quad (3.3)$$

This limit is well define since  $\mathcal{D}_n \subset \mathbb{L}_b$ , and

$$p_m(i, j) g_i(\eta) (\varphi(T_j^i \eta) - \varphi(\eta)) \leq L(\varphi) p_m(i, j) g_i(\eta) (\alpha(i) + \alpha(j)),$$

so that by Lemma 2.1 of [13], we have that

$$\sum_{k \geq 0} \int (\mathcal{L}_{(m)}^k(\varphi))^2 1_{\Omega_{(m)}^k} d\nu_\rho < \infty.$$

Also, the expression  $\mathcal{L}_{(m)}^k(\varphi)$ , and the limit (3.3) are independent of  $m$  when  $m > n + R$ , and we called the latter  $E_{\nu_\rho}[\mathcal{L}(\varphi) | \mathcal{F}_{\Lambda_n}]$  in the Introduction. Since  $\{\mathcal{L}_{(m)}^k, k \in \mathbb{N}\}$  have the same expression, we henceforth drop the index  $k$ , as well as  $\rho$  in  $\mathcal{L}_n^\rho$  since we work with a fixed density  $\rho > 0$ . Finally, a simple computation gives an expression for  $\mathcal{L}_n$

$$\mathcal{L}_n(\varphi) = \mathcal{L}_{(n)}(\varphi) + \sum_{i \in \Lambda_n} p_n^*(i, i) \gamma_\rho(\varphi \circ \mathbf{A}_i^+ - \varphi) + \sum_{i \in \Lambda_n} p_n(i, i) g_i(\varphi \circ \mathbf{A}_i^- - \varphi). \quad (3.4)$$

Note that by definition of  $\mathcal{L}_n$ , the product of measures  $\theta_{\gamma(\rho)}$  over sites of  $\Lambda_n$ , which we denote either by  $\nu_\rho^{\Lambda_n}$  or simply by  $\nu_\rho$ , is the invariant measure for  $\mathcal{L}_n$ . Also, we have

$\mathcal{L}_n^*(\varphi) = E_{\nu_\rho}[\mathcal{L}^*(\varphi)|\mathcal{F}_{\Lambda_n}]$ . Finally, we omit the simple proof that  $\mathcal{L}_n$  is a monotone irreducible process.

We denote by  $E_\eta^n$  (resp.  $E_\eta^{(n)}$ ) the law of the Markov process generated by  $\mathcal{L}_n$  (resp.  $\mathcal{L}_{(n)}$ ) with initial configuration  $\eta$ . We denote by  $\tilde{\mathcal{L}}_n := 1_{\mathcal{A}^c}\mathcal{L}_n$  (resp.  $\tilde{\mathcal{L}}_{(n)} := 1_{\mathcal{A}^c}\mathcal{L}_{(n)}$ ) the process killed on  $\mathcal{A}$ , and by  $\bar{S}_t^n$  (resp.  $\bar{S}_t^{(n)}$ ) the associated semi-group. Note that if  $\tau$  is the first occurrence time of  $\mathcal{A}$ , then for  $\varphi|_{\mathcal{A}} \equiv 0$

$$\bar{S}_t^n(\varphi)(\eta) = E_\eta^n[\varphi(\eta_{t \wedge \tau})] = E_\eta^n[\varphi(\eta_t)1_{\tau > t}].$$

### 3.2 Approximating the killed process.

The main uniqueness result is the following.

**Lemma 3.1** *For any  $\varphi \in \mathbb{L}_b$  with  $\varphi|_{\mathcal{A}} \equiv 0$ , we have*

$$\forall t > 0, \quad \lim_{n \rightarrow \infty} \int |\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)| d\nu_\rho = 0.$$

*Proof.* We first approximate  $\{\tau > t\}$  by  $\{\eta(t_i) \notin \mathcal{A}, i = 0, \dots, k\}$  where  $\{t_i\}$  is a regular subdivision of  $[0, t]$  of mesh  $t/k$ ; we denote the latter event  $\{\tau^k > t\}$ . Thus, we show in Step 1 that for each  $k > 0$ , and  $\varphi \in \mathbb{L}_b$  with  $\varphi|_{\mathcal{A}} \equiv 0$

$$\lim_{n \rightarrow \infty} \int |E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k > t\}}\varphi(\eta_t)]| d\nu_\rho = 0. \quad (3.5)$$

Since by Lemmas 2.3 and 2.6 of [1], we have the pointwise convergence

$$E_\eta^{(n)}[1_{\{\tau^k > t\}}\varphi(\eta_t)] = S_{t_1}^{(n)} \left( 1_{\mathcal{A}^c} S_{t_2}^{(n)} \left( 1_{\mathcal{A}^c} \dots S_{t_{k+1}}^{(n)}(\varphi) \right) \right) (\eta) \xrightarrow{n \rightarrow \infty} E_\eta[1_{\{\tau^k > t\}}\varphi(\eta_t)], \quad (3.6)$$

we would conclude that

$$\lim_{n \rightarrow \infty} \int |E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] - E_\eta[1_{\{\tau^k > t\}}\varphi(\eta_t)]| d\nu_\rho = 0. \quad (3.7)$$

In Step 2, we show that there is a constant  $C$  independent of  $n$  such that

$$\int |E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] - E_\eta^n[1_{\{\tau > t\}}\varphi(\eta_t)]| d\nu_\rho \leq C\epsilon. \quad (3.8)$$

Also, leaving  $\mathcal{A}$  requires that all the particles in *excess* escape  $\mathcal{S}$  in a subinterval of length  $t/k$ . Thus, the continuity properties of the infinite volume process give

$$\lim_{k \rightarrow \infty} E_\eta[1_{\{\tau^k > t\}}\varphi(\eta_t)] = E_\eta[1_{\{\tau > t\}}\varphi(\eta_t)], \quad (3.9)$$

and the proof is concluded once we combine (3.7), (3.8) and (3.9).

**Step 1.**

First, we show by induction on  $k$  (the number of points in the subdivision of  $[0, t]$ ) that there are two constants  $C_k, C'_k$  such that for  $\eta \notin \mathcal{A}$  if we set  $\delta_n(i) = (p_n(i, i) + p_n^*(i, i))\alpha(i)$

$$|E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k > t\}}\varphi(\eta_t)]| \leq C_k \sum_{i \in \Lambda_n} \delta_n(i) \sum_{j=0}^{k-1} \int_0^t E_\eta^n[\eta_{s+s_j}(i) + C'_k] ds, \quad (3.10)$$

where  $s_0 = 0$  and  $s_j = t_1 + \dots + t_j$ .

For  $k = 1$ , we have  $t_0 = 0$  and  $t_1 = t$ , so that (3.10) reduces to show that for  $\eta \notin \mathcal{A}$ , there are  $C_1, C'_1$  such that

$$|S_t^n \varphi(\eta) - S_t^{(n)} \varphi(\eta)| \leq C_1 \sum_{i \in \Lambda_n} \delta_n(i) \int_0^t E_\eta^n[\eta_s(i) + C'_1] ds, \quad (3.11)$$

To obtain (3.11), we use an integration by parts formula

$$S_t^n \varphi(\eta) - S_t^{(n)} \varphi(\eta) = \int_0^t S_{t-s}^n (\mathcal{L}_n - \mathcal{L}_{(n)}) S_s^{(n)} \varphi(\eta) ds.$$

Since  $\varphi \in \mathbb{L}_b$ , Lemma 2.2 of [1] implies that for some constant  $C$

$$L(S_s^{(n)} \varphi) \leq e^{Cs} L(\varphi).$$

From (3.4) it is enough to bound terms of the form

$$|A_i^\pm S_s^{(n)} \varphi(\eta) - S_s^{(n)} \varphi(\eta)| \leq L(S_s^{(n)} \varphi) \alpha(i) \leq L(\varphi) e^{Cs} \alpha(i). \quad (3.12)$$

Thus,

$$|S_t^n \varphi(\eta) - S_t^{(n)} \varphi(\eta)| \leq L(\varphi) \sum_{i \in \Lambda_n} \delta_n(i) \int_0^t (S_s^n(g_i)(\eta) + \gamma_\rho) ds.$$

(3.11) follows after recalling that  $g_i(\eta) \leq \Delta \eta(i)$ .

The induction step from  $k$  to  $k+1$  follows with exactly the same arguments. First, we recall (3.6) and write similarly

$$E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] = S_{t_1}^n \left( 1_{\mathcal{A}^c} S_{t_2}^n \left( 1_{\mathcal{A}^c} \dots S_{t_{k+1}}^n(\varphi) \right) \right).$$

We call  $\psi_2 := S_{t_2}^{(n)}(1_{\mathcal{A}^c} S_{t_3}^{(n)}(1_{\mathcal{A}^c} \dots))$ , and recall that  $\psi_2 \in \mathbb{L}_b$  by Lemma 2.3 of [1]. We now show that  $1_{\mathcal{A}^c} \psi_2 \in \mathbb{L}_b$ . Indeed, for  $\eta, \zeta \in \Omega$

$$|\psi_2(\eta) 1_{\eta \in \mathcal{A}^c} - \psi_2(\zeta) 1_{\zeta \in \mathcal{A}^c}| \leq 1_{\eta, \zeta \in \mathcal{A}^c} |\psi_2(\eta) - \psi_2(\zeta)| + 1_B(\eta, \zeta) |\psi_2|_\infty. \quad (3.13)$$

where we set  $B := \mathcal{A} \times \mathcal{A}^c \cup \mathcal{A}^c \times \mathcal{A}$ . Now,  $(\eta, \zeta) \in B$  implies that  $\sum_{\mathcal{S}} |\eta(i) - \zeta(i)| \geq 1$ . Thus,

$$1_B(\eta, \zeta) \leq \sum_{\mathcal{S}} |\eta(i) - \zeta(i)| \leq \frac{\sum_{\mathcal{S}} |\eta(i) - \zeta(i)| \alpha(i)}{\inf_{\mathcal{S}} \alpha(i)} \leq C \|\eta - \zeta\|. \quad (3.14)$$

Thus, combining (3.14) and (3.13) we obtain that  $1_{\mathcal{A}^c}\psi \in \mathbb{L}_b$ . Now,

$$\begin{aligned} E_\eta^n[1_{\{\tau^k > t\}}\varphi(\eta_t)] - E_\eta^{(n)}[1_{\{\tau^k > t\}}\varphi(\eta_t)] &= \left( S_{t_1}^n(1_{\mathcal{A}^c}\psi_2) - S_{t_1}^{(n)}(1_{\mathcal{A}^c}\psi_2) \right) \\ &\quad - S_{t_1}^n(1_{\mathcal{A}^c}(\psi_2 - S_{t_2}^n(1_{\mathcal{A}^c}S_{t_3}^n(1_{\mathcal{A}^c}\dots)))) \end{aligned} \quad (3.15)$$

To the first term on the r.h.s we apply the estimates of the step  $k = 1$  of the induction. For the second term, the difference  $\psi_2 - S_{t_2}^n(1_{\mathcal{A}^c}S_{t_3}^n(1_{\mathcal{A}^c}\dots))$  has  $k$  subdivision times, and we use our induction hypothesis to obtain (3.10) at order  $k$ ; since  $S_{t_1}^n$  is positive preserving, the inequality is preserved after applying  $S_{t_1}^n$  and we obtain the desired (3.10) at order  $k + 1$ . Now, to obtain (3.6), note that

$$\sum_{i \in \Lambda_n} \delta_n(i) \leq C \sum_{i \in \Lambda_n \setminus \Lambda_{n-R}} \alpha(i) \xrightarrow{n \rightarrow \infty} 0 \quad (\text{since } \sum_{i \in \mathbb{Z}^d} \alpha(i) < \infty).$$

**Step 2.** Let  $\sigma_{\mathcal{S}}$  be the first time a particle inside  $\mathcal{S}$  escapes  $\mathcal{S}$ , and let  $\theta_t$  be the time-translation by  $t$ . By the strong Markov property, for  $\eta \notin \mathcal{A}$  and  $\epsilon = t/k$

$$\begin{aligned} |P_\eta^n(\tau > t) - P_\eta^n(\tau^k > t)| &\leq P_\eta^n \left( \bigcup_{i \leq k} \{\tau \in [t_{i-1}, t_i[, \sigma_{\mathcal{S}} \circ \theta_\tau < \epsilon\} \right) \\ &= \sum_{i=1}^k E_\eta^n[1_{\tau \in [t_{i-1}, t_i[} P_{\eta_\tau}^n(\sigma_{\mathcal{S}} < \epsilon)]. \end{aligned} \quad (3.16)$$

We need now a uniform estimate on  $P_{\eta_\tau}^n(\sigma_{\mathcal{S}} < \epsilon) \leq C\epsilon$ . By the hypotheses made on  $\mathcal{A}$ , we know that at time  $\tau$ , there is a bounded number of particles in  $\mathcal{S}$ . For the zero range process, it is routine to couple, from time  $\tau$  onward, the motion of the particle inside  $\mathcal{S}$  (at time  $\tau$ ) with a process containing only particles in  $\mathcal{S}$  distributed as those of  $\eta_\tau$ . Now, for this new process, at any site, the rate of jump is bounded (uniformly in  $\eta_\tau$ , since the number of particles is uniformly bounded), and the probability of having a jump before time  $\epsilon$  is smaller than  $1 - \exp(-\bar{c}\epsilon) \leq \bar{c}\epsilon$ . This concludes Step 2.  $\blacksquare$

### 3.3 Donsker-Varadhan functionals in $\Lambda_n$

For  $(\varphi, \mu) \in \mathcal{D}_n^+ \times \mathcal{M}_n$ , we define

$$\Gamma_n(\varphi, \mu) := \int \frac{\mathcal{L}_n \varphi}{\varphi} d\mu. \quad (3.17)$$

This is well defined since  $\varphi > 0$  on  $\mathcal{A}^c$  which contains the support of  $\mu$ . The functional  $\Gamma_n(\varphi, \mu)$  is useful if it has some regularity in  $\mu$  and convexity in  $\log(\varphi)$ .

**Lemma 3.2** *Assume  $d \geq 3$ . (i) For any  $\varphi \in \mathcal{D}_n$ ,  $\Gamma_n(\varphi, \cdot) : \mathcal{M}_n \rightarrow \mathbb{R}$  is continuous. (ii) For any  $\mu \in \mathcal{M}_n$ , the map  $\tilde{\Gamma}_n(\cdot, \mu) := \Gamma_n(\exp(\cdot), \mu) : \mathcal{E}_n \rightarrow \mathbb{R}$  is convex.*

*Proof.* Since  $\mathcal{L}_n(\varphi)/\varphi$  is not bounded, point (i) is not obvious. Let  $\{\mu_k, k \in \mathbb{N}\}$  be in  $\mathcal{M}_n$  converging weakly to  $\mu$ . We show that for any  $\varphi \in \mathcal{D}_n$ ,  $\Gamma_n(\varphi, \mu_k)$  converges to  $\Gamma_n(\varphi, \mu)$  as  $k$  tends to infinity. We recall the notation  $\nabla_j^i = T_j^i - 1$ ,

$$\Gamma_n(\varphi, \mu_k) := \sum_{i,j \in \Lambda_n} p(i, j) \int g_i \frac{\nabla_j^i \varphi}{\varphi} d\mu_k + \sum_{i \in \Lambda_n} \int (p_n^*(i, i) \gamma_\rho \frac{A_i^+ \varphi - \varphi}{\varphi} + p_n(i, i) g_i \frac{A_i^- \varphi - \varphi}{\varphi}) d\mu_k. \quad (3.18)$$

Let  $K_M$  be the compact set defined in (2.16). When integrating over  $K_M$ , the integrals on the r.h.s of (3.18) pose no problem since the integrant over  $K_M$  is bounded. When integrating over  $K_M^c$ , first we recall that by Lemma 2.4, we have that  $\varphi, 1_{\mathcal{A}^c}/\varphi, g_i$  as well as  $f_k := d\mu_k/(d\nu_\rho)$  are in  $L^p(\nu_\rho)$  for any  $p \geq 1$ . We then use Hölder's inequality for  $p = 5$

$$\begin{aligned} \int_{K_M^c} g_i \frac{T_j^i \varphi}{\varphi} d\mu_k &\leq \int_{K_M^c} g_i \frac{\varphi \circ A_i^-}{\varphi} f_k d\nu_\rho \\ &\leq \left( \int g_i \varphi^p \circ A_i^- d\nu_\rho \int \frac{1_{\mathcal{A}^c}}{\varphi^p} d\nu_\rho \int f_k^p d\nu_\rho \int g_i^{p-1} d\nu_\rho \nu_\rho(K_M^c) \right)^{1/p} \\ &\leq \left( \gamma_\rho \int \varphi^p d\nu_\rho \int \frac{1_{\mathcal{A}^c}}{\varphi^p} d\nu_\rho \int f_k^p d\nu_\rho \int g^{p-1} d\nu_\rho \right)^{1/p} \nu_\rho(K_M^c)^{1/p} \\ &\leq C \nu_\rho(K_M^c)^{1/p} \xrightarrow{M \rightarrow \infty} 0. \end{aligned} \quad (3.19)$$

The other terms of (3.18) are dealt with in the same way. To establish (ii), note first that by Lemma 2.2,  $\mathcal{E}_n$  is convex. Then

$$\begin{aligned} \Gamma_n(e^h, \mu) &= \sum_{i,j \in \Lambda_n} p(i, j) \int g_i (e^{\nabla_j^i h} - 1) d\mu \\ &\quad + \sum_{i \in \Lambda_n} p_n^*(i, i) \gamma_\rho \int (e^{h \circ A_i^+ - h} - 1) + p_n(i, i) \int g_i (e^{h \circ A_i^- - h} - 1) d\mu. \end{aligned} \quad (3.20)$$

The convexity follows from the convexity of the exponential. ■

### 3.4 A variational formula for $\lambda_n(\rho)$ .

**Lemma 3.3** *For  $d \geq 1$ , there is  $u_n \in \mathcal{D}_n$  and  $\lambda_n(\rho) > 0$  such that*

$$1_{\mathcal{A}^c} \mathcal{L}_n(u_n) + \lambda_n(\rho) u_n = 0. \quad (3.21)$$

*Moreover  $u_n$  is positive on  $\mathcal{A}^c$ .*

*Similarly, when  $d \geq 1$ , there is  $u_n^* \in \mathcal{D}_n^*$ , positive on  $\mathcal{A}^c$ , which satisfies  $1_{\mathcal{A}^c} \mathcal{L}_n^* u_n^* + \lambda_n(\rho) u_n^* = 0$ , and*

$$-\lambda_n(\rho) = \lim_{t \rightarrow \infty} \frac{1}{t} \log(P_{\nu_\rho}^n(\tau > t)). \quad (3.22)$$

*Proof.* The proof follows the same lines as that of [2] (see also [8]). This is expected since  $\mathcal{L}_n$  is a monotone operator with the same features as  $\mathcal{L}$ . Thus, (3.22) follows as simply as

(1.3) by a subadditivity argument. Now, for  $\eta \in \Omega_n$ , we denote

$$u_{t,n}(\eta) = \frac{P_\eta^n(\tau > t)}{P_{\nu_\rho}^n(\tau > t)} = \frac{e^{t1_{\mathcal{A}^c}\mathcal{L}_n(1_{\mathcal{A}^c})(\eta)}}{P_{\nu_\rho}^n(\tau > t)}, \text{ and } u_{t,n}^*(\eta) = \frac{e^{t1_{\mathcal{A}^c}\mathcal{L}_n^*(1_{\mathcal{A}^c})(\eta)}}{P_{\nu_\rho}^n(\tau > t)} \quad (3.23)$$

and as in Step 1 of the proof of Lemma 2.6 of [2],  $u_{t,n} \in \mathcal{D}_n$  and  $u_{t,n}^* \in \mathcal{D}_n^*$ . We focus now on  $u_{t,n}$ , though similar properties will hold for  $u_{t,n}^*$ . First, by Lemma 1.1,  $\lambda_n(\rho) \geq \lambda(\rho) > 0$ . Thus, for any  $k$ ,  $\int_0^\infty P_{\nu_\rho}^n(\tau > t)t^k dt < \infty$ , and as in Remark 2.7 we define

$$dm_k(t) = \frac{P_{\nu_\rho}^n(\tau > t)t^k dt}{\int_0^\infty P_{\nu_\rho}^n(\tau > t)t^k dt} \quad \text{and} \quad \frac{d\Phi_n^{(k)}(\nu_\rho)}{d\nu_\rho}(\eta) = \int_0^\infty u_{t,n}(\eta) dm_k(t).$$

With identical arguments as in the proof of Theorem 2.4 of [2], the Cesaro weak- $L^2(\nu_\rho)$  limits of  $\{\Phi_n^{(k)}(\nu_\rho), k \in \mathbb{N}\}$  are solutions of (3.21). Now, it is clear that  $d\Phi_n^{(k)}(\nu_\rho)/d\nu_\rho \in \mathcal{D}_n$ . Also, in the weak- $L^2(\nu_\rho)$  topology  $\mathcal{D}_n$  is compact by Remark 2.6, and contain all the Cesaro weak limits of  $\{\Phi_n^{(k)}(\nu_\rho), k \in \mathbb{N}\}$ . Thus, there is a solution of (3.21) in  $\mathcal{D}_n$ : we denote it by  $u_n$ .

We now show that  $u_n > 0$  on  $\mathcal{A}^c$ . By contradiction assume that for  $\eta \in \Omega_n \setminus \mathcal{A}$ ,  $u_n(\eta) = 0$ . Then (3.22) implies that  $\mathcal{L}_n(u_n)(\eta) = 0$ . This, in turn, implies that

- (i) For all  $i, j \in \Lambda_n$  with  $p(i, j) > 0$ , we have  $u_n(T_j^i \eta) = 0$ .
- (ii) For all  $i \in \Lambda_n$  with  $p_n^*(i, i) > 0$ , we have  $u_n(\mathbf{A}_i^+ \eta) = 0$ .
- (iii) For all  $i \in \Lambda_n$  with  $\eta(i)p_n(i, i) > 0$ , we have  $u_n(\mathbf{A}_i^- \eta) = 0$ .

To conclude that  $u_n \equiv 0$  on  $\mathcal{A}^c$ , it is enough to note that by the hypotheses  $(\mathcal{C}-\mathcal{F})$  on  $\mathcal{A}^c$ , each  $\eta \in \mathcal{A}^c$  can be transformed into  $O_{\Lambda_n}$  by a succession of actions  $\{\mathbf{A}_i^-\}$  with  $i \in \Lambda_n$ , and  $\{T_j^i\}$  with  $i, j \in \Lambda_n$ . The reverse operation is made through a succession of  $\{\mathbf{A}_i^+\}$  with  $i \in \Lambda_n$ , and  $\{T_j^i\}$  with  $i, j \in \Lambda_n$ .  $\blacksquare$

We now establish the Donsker-Varadhan representation for  $\lambda_n(\rho)$ .

**Lemma 3.4** *Assume  $d \geq 3$ . If  $\mathcal{A}$  satisfies  $(\mathcal{C}-\mathcal{F})$  of Section 2.2, then  $\lambda_n(\rho)$  is given by*

$$-\lambda_n(\rho) = \sup_{\mu \in \mathcal{M}_n} \inf_{\varphi \in \mathcal{D}_n^+} \int \frac{\mathcal{L}_n \varphi}{\varphi} d\mu. \quad (3.24)$$

*Proof.* Let us call  $\gamma_n$  the right hand side of (3.24). From Lemma 3.3, there is  $u_n \in \mathcal{D}_n^+$  such that  $\bar{\mathcal{L}}_n u_n + \lambda_n(\rho) u_n = 0$ . This implies that  $\gamma_n \leq -\lambda_n(\rho)$ . We can use a classical minimax theorem [7], since we have that (i) for any fixed  $\mu \in \mathcal{M}_n$ ,  $h \mapsto \tilde{\Gamma}_n(h, \mu)$  is convex (by Lemma 3.2) on the convex set  $\mathcal{E}_n$  (by Lemma 2.2), (ii) for any fixed  $h \in \mathcal{E}_n$ ,  $\mu \mapsto \tilde{\Gamma}_n(h, \mu)$  is continuous (by Lemma 3.2) on the compact set  $\mathcal{M}_n$ . Thus,

$$\gamma_n = \inf_{\varphi \in \mathcal{D}_n^+} \sup_{\mu \in \mathcal{M}_n} \int \frac{\mathcal{L}_n \varphi}{\varphi} d\mu. \quad (3.25)$$

Now, for any  $\varphi \in \mathcal{D}_n^+$ ,  $0 < \int \varphi u_n^* d\nu_\rho < \infty$ , and we can define

$$d\mu^* = \frac{\varphi u_n^* d\nu_\rho}{\int \varphi u_n^* d\nu_\rho} \in \mathcal{M}_n \quad (\text{by Lemma 2.4}).$$



Then, by duality

$$\int \frac{\mathcal{L}_n(\varphi)}{\varphi} d\mu^* = \int \frac{\mathcal{L}_n(\varphi)}{\varphi} \frac{\varphi u_n^*}{\int \varphi u_n^* d\nu_\rho} d\nu_\rho = \int \frac{\varphi}{\int \varphi u_n^* d\nu_\rho} \mathcal{L}_n^*(u_n^*) d\nu_\rho = -\lambda_n(\rho).$$

By (3.25),  $\gamma_n \geq -\lambda_n(\rho)$ , and the proof is concluded.  $\blacksquare$

In the following lemma, we establish the uniqueness of the principal Dirichlet eigenfunction.

**Lemma 3.5** *Assume  $d \geq 3$ . There is a unique non-negative eigenfunction  $u_n \in \mathcal{D}_n$  of  $1_{\mathcal{A}^c} \mathcal{L}_n$  which satisfies  $\int u_n d\nu_\rho = 1$ .*

*Proof.* We know from Lemma 3.3 that there exists a positive eigenfunction  $u_n$ . Assume that  $\tilde{u}$  is a non-negative Dirichlet eigenfunction with  $\int \tilde{u} d\nu_\rho = 1$  and corresponding eigenvalue  $\tilde{\lambda}$ . By the same argument as in the proof of Lemma 3.3, we have that  $\tilde{u}$  is positive on  $\mathcal{A}^c$ .

First, we show that  $\tilde{\lambda} = \lambda_n$ . Let  $u_n^*$  be the dual eigenfunction given in Lemma 3.3. We multiply equality (3.21) by  $u_n^*$ , integrate over  $\nu_\rho$  and use duality

$$\int u_n^* \mathcal{L}_n(\tilde{u}) d\nu_\rho = -\tilde{\lambda} \int u_n^* \tilde{u} d\nu_\rho \implies (\lambda_n(\rho) - \tilde{\lambda}) \int u_n^* \tilde{u} d\nu_\rho = 0. \quad (3.26)$$

Now, since  $u_n^*$  and  $\tilde{u}$  are positive on  $\mathcal{A}^c$  we conclude that  $\tilde{\lambda} = \lambda_n(\rho)$ .

Second, we show that  $\tilde{u} = u_n$ . Set  $h := \log(u_n)$  and  $\tilde{h} := \log(\tilde{u})$ , on  $\mathcal{A}^c$ . For any  $\mu \in \mathcal{M}_n$  and any  $\gamma \in ]0, 1[$ , by the convexity of  $\tilde{\Gamma}_n$

$$\gamma \tilde{\Gamma}_n(h, \mu) + (1 - \gamma) \tilde{\Gamma}_n(\tilde{h}, \mu) \geq \tilde{\Gamma}_n(\gamma h + (1 - \gamma) \tilde{h}, \mu). \quad (3.27)$$

Since  $u_n$  and  $\tilde{u}$  are solution of (3.21), the left hand side of (3.27) is  $-\lambda_n(\rho)$ . We define  $h_\gamma = \gamma h + (1 - \gamma) \tilde{h} \in \mathcal{E}_n$  and we note that  $0 < \int \exp(h_\gamma) u_n^* d\nu_\rho < \infty$ . Now,

$$d\mu_\gamma = \frac{e^{h_\gamma} u_n^* d\nu_\rho}{\int e^{h_\gamma} u_n^* d\nu_\rho} \in \mathcal{M}_n, \quad \text{and is such that} \quad \tilde{\Gamma}_n(h_\gamma, \mu_\gamma) = \Gamma_n(u_n^*, \mu_\gamma) = -\lambda_n(\rho).$$

Thus, we have equality in (3.27) with  $\mu_\gamma$ . Since  $\mu_\gamma$  gives a positive weight to any  $\eta \in \Omega_n \setminus \mathcal{A}$ , the following three conditions hold: (i) for all  $i, j \in \Lambda_n$  with  $g_i(\eta) p(i, j) > 0$ , we have  $\nabla_j^i \tilde{h} = \nabla_j^i h$ ; (ii) for all  $j \in \Lambda_n$  with  $p_n^*(j, j) > 0$ , we have  $(\mathbf{A}_j^+ - \mathbf{1}) \tilde{h} = (\mathbf{A}_j^+ - \mathbf{1}) h$ ; (iii) for all  $j \in \Lambda_n$  with  $g(\eta(j)) p_n(j, j) > 0$ , we have  $(\mathbf{A}_j^- - \mathbf{1}) \tilde{h} = (\mathbf{A}_j^- - \mathbf{1}) h$ .

Since  $u_n$  is positive on  $\mathcal{A}^c$ , we form  $f = \tilde{u}/u_n$ , and rewrite the conditions (i)-(iii) for  $f$ .

- (i) For all  $\eta \in \mathcal{A}^c$  and  $i, j \in \Lambda_n$  with  $\eta(i) p(i, j) > 0$ , we have  $f(T_j^i \eta) = f(\eta)$ .
- (ii) For all  $i \in \Lambda_n$  with  $p_n^*(i, i) > 0$ , and  $\mathbf{A}_i^+ \eta \in \mathcal{A}^c$ , we have  $f(\mathbf{A}_i^+ \eta) = f(\eta)$ .
- (iii) For all  $\eta \in \mathcal{A}^c$  and  $i \in \Lambda_n$  with  $\eta(i) p_n(i, i) > 0$ , we have  $f(\mathbf{A}_i^- \eta) = f(\eta)$ .

As in the proof of Lemma 3.3, we conclude that  $\tilde{u} = u_n$ .  $\blacksquare$

### 3.5 Approximating the principal eigenvalue.

With an abuse of notations, we define for any finite domain  $U$ ,  $\mathcal{L}_U(\varphi) = E_{\nu_\rho}[\mathcal{L}(\varphi)|\mathcal{F}_U]$ . We mean by  $\mathcal{L}_U$  an expression like (3.4) where  $U$  replaces  $\Lambda_n$ : thus, a zero-range process on  $U$  with creations and annihilations on the boundaries of  $U$ . We denote by  $S_t^U$  the semi-group associated with  $\mathcal{L}_U$  and by  $P_\nu^U$  the corresponding Markov process with initial measure  $\nu$ . We denote by  $\bar{S}_t^U$  the semi-group killed on  $\mathcal{A}$ .

We first state an obvious corollary of Lemma 3.1 applied to  $\varphi = 1_{\mathcal{A}^c}$ .

**Corollary 3.6** *When the pattern satisfies  $(\mathcal{C}-\mathcal{F})$ , we have*

$$\lim_{n \rightarrow \infty} P_{\nu_\rho}^n(\tau > t) = P_{\nu_\rho}(\tau > t).$$

**Proof of Lemma 1.1** We divide the proof in two steps.

**Step 1.** We show that  $n \mapsto P_{\nu_\rho}^n(\tau > t)$  is increasing.

Let  $U$  be a finite subset,  $i \notin U$ , and set  $\tilde{U} = U \cup \{i\}$ . Thus, it is enough to show that  $\int (\bar{S}_t^{\tilde{U}} 1_{\mathcal{A}^c} - \bar{S}_t^U 1_{\mathcal{A}^c}) d\nu_\rho \geq 0$ . Step 1 follows then by induction. Note that for  $\varphi \mathcal{F}_U$ -measurable and  $j \in U$ , we have  $\varphi \circ T_j^i = \varphi \circ \mathbf{A}_j^+$ ,  $\varphi \circ T_i^j = \varphi \circ \mathbf{A}_j^-$ ,  $\varphi \circ \mathbf{A}_i^+ = \varphi$  and  $\varphi \circ \mathbf{A}_i^- = \varphi$  so that

$$\begin{aligned} (\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)\varphi &= 1_{\mathcal{A}^c} \sum_{j \in U} (p(j, i)g_j(\varphi \circ T_i^j - \varphi) + p(i, j)g_i(\varphi \circ T_j^i - \varphi)) \\ &\quad - 1_{\mathcal{A}^c} \sum_{j \in U} (p(j, i)g_j(\varphi \circ \mathbf{A}_j^- - \varphi) + p(i, j)\gamma_\rho(\varphi \circ \mathbf{A}_j^+ - \varphi)) \\ &= 1_{\mathcal{A}^c} \sum_{j \in U} p(i, j)(g_i - \gamma_\rho)(\varphi \circ \mathbf{A}_j^+ - \varphi). \end{aligned} \quad (3.28)$$

Now, we set  $\varphi_s := \bar{S}_s^U(1_{\mathcal{A}^c})$  and  $\psi_s := (\bar{S}_s^{\tilde{U}})^*(1_{\mathcal{A}^c})$ , and we use an integration by parts formula

$$\begin{aligned} \int \bar{S}_t^{\tilde{U}}(1_{\mathcal{A}^c}) d\nu_\rho - \int \bar{S}_t^U(1_{\mathcal{A}^c}) d\nu_\rho &= \int \int_0^t \bar{S}_{t-s}^{\tilde{U}}(\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)\bar{S}_s^U(1_{\mathcal{A}^c}) ds d\nu_\rho \\ &= \int \int_0^t (\bar{\mathcal{L}}_{\tilde{U}} - \bar{\mathcal{L}}_U)(\varphi_s)\psi_{t-s} ds d\nu_\rho. \end{aligned} \quad (3.29)$$

Thus, by (3.28)

$$\begin{aligned} P_{\nu_\rho}^{\tilde{U}}(\tau > t) - P_{\nu_\rho}^U(\tau > t) &= \sum_{j \in U} p(i, j) \int \int_0^t (\mathbf{A}_j^+ \varphi_s - \varphi_s)(g_i - \gamma_\rho)\psi_{t-s} ds d\nu_\rho^{\tilde{U}} \\ &= \sum_{j \in U} p(i, j) \int \int_0^t (\mathbf{A}_j^+ \varphi_s - \varphi_s) \int (g_i - \gamma_\rho)\psi_{t-s} d\nu_\rho^{\{i\}} ds d\nu_\rho^U. \end{aligned} \quad (3.30)$$

Note that for any  $s$ ,  $\eta \mapsto \psi_s(\eta)$  and  $\eta \mapsto \varphi_s(\eta)$  is decreasing positive, whereas  $\eta \mapsto g_i(\eta)$  is increasing and  $\int g_i d\nu_\rho = \gamma_\rho$ . Thus, by FKG inequality

$$\int (g_i - \gamma_\rho)\psi_{t-s} d\nu_\rho^{\{i\}} \leq \int (g_i - \gamma_\rho) d\nu_\rho^{\{i\}} \int \psi_{t-s} d\nu_\rho^{\{i\}} = 0. \quad (3.31)$$

Thus, as  $\varphi_s \circ \mathbf{A}_j^+ - \varphi_s \leq 0$ , the first step concludes. We call  $\lambda_\infty(\rho)$  the limit of  $\lambda_n(\rho)$ .

**Step 2.** We show the following Lemma which allows us to conclude the proof of Lemma 1.1 readily.

**Lemma 3.7** *Any subsequence of  $\{u_n\}$  has a further subsequence converging, in weak- $L^2(\nu_\rho)$ , to a solution  $u$  of (1.4(i)), and  $u \in \mathcal{D}_\rho$ . Moreover,  $\lambda_\infty(\rho) = \lambda(\rho)$ .*

*Proof.* For notational convenience, we write the proof for  $\{u_n^*\}$ . Recall that  $\mathcal{D}_\rho^* \cap \{\varphi : \int \varphi d\nu_\rho = 1\}$  is compact in weak- $L^2(\nu_\rho)$  by Remark 2.6. Let  $u^* \in \mathcal{D}_\rho^*$  be a limit point of  $\{u_n^*\}$  along a subsequence which for simplicity we still call  $\{u_n^*\}$ . For any  $\varphi \in \mathbb{L}_b$ , and any integer  $n$

$$\int \bar{S}_t^n(\varphi) u_n^* d\nu_\rho = e^{-\lambda_n(\rho)t} \int \varphi u_n^* d\nu_\rho. \quad (3.32)$$

Then,

$$\begin{aligned} \left| \int \bar{S}_t^n(\varphi) u_n^* d\nu_\rho - \int \bar{S}_t(\varphi) u^* d\nu_\rho \right| &= \left| \int (\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)) u_n^* d\nu_\rho \right| + \left| \int \bar{S}_t(\varphi) (u_n^* - u^*) d\nu_\rho \right| \\ &\leq \sup_n \|u_n^*\|_{\nu_\rho} \|\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)\|_{\nu_\rho} + \left| \int \bar{S}_t(\varphi) (u_n^* - u^*) d\nu_\rho \right|. \end{aligned} \quad (3.33)$$

The  $L^2(\nu_\rho)$  convergence of  $\bar{S}_t^n(\varphi) - \bar{S}_t(\varphi)$  is equivalent to an  $L^1(\nu_\rho)$  convergence, since  $\varphi$  is bounded and  $\bar{S}_t, \bar{S}_t^n$  are contractions (in  $L^\infty$ ). By recalling Lemma 3.1 and Step 1, and taking the limit  $n$  to infinity,

$$\int \bar{S}_t(\varphi) u^* d\nu_\rho = e^{-\lambda_\infty(\rho)t} \int \varphi u^* d\nu_\rho. \quad (3.34)$$

Now, since  $\mathbb{L}_b$  is a dense set in  $L^2(\nu_\rho)$ , this implies that  $u^* \in \mathcal{D}(\bar{\mathcal{L}}^*, L^2(\nu_\rho))$ , and that (3.34) holds for any  $\varphi \in L^2(\nu_\rho)$ . Take  $\varphi = u \in \mathcal{D}_\rho \subset L^2(\nu_\rho)$  solution of (1.4(i)), and use that

$$\bar{S}_t(u) = e^{-\lambda(\rho)t} u, \quad \nu_\rho - \text{a.s.} \implies (e^{-\lambda_\infty(\rho)t} - e^{-\lambda(\rho)t}) \int u u^* d\nu_\rho = 0.$$

Now, since  $u$  and  $u^*$  are decreasing, and in  $L^2(\nu_\rho)$ , we have

$$\infty > \|u\|_{\nu_\rho} \|u^*\|_{\nu_\rho} \geq \int u u^* d\nu_\rho \stackrel{\text{FKG}}{\geq} \int u d\nu_\rho \int u^* d\nu_\rho = 1.$$

Thus,  $\lambda_\infty(\rho) = \lambda(\rho)$ , and  $u^*$  satisfies (1.4(ii)). ■

## 4 Donsker-Varadhan functionals on $\mathbb{Z}^d$

The main problem arises since  $\mathcal{L}(\varphi)$  does not make sense as a pointwise convergent series when  $\varphi \in \mathcal{D}_\rho$ . Indeed, even if  $\varphi$  were bounded, the naive bound  $|\nabla_j^i \varphi| \leq |\varphi|_\infty (\epsilon_i + \epsilon_j)$  would fail since  $\sum_k \epsilon_k = \infty$ . Thus, we show in this section how to obtain  $\Gamma_\infty(\varphi, \mu)$  as the limit of the Cauchy sequence  $\{\int \mathcal{L}_n^\rho(\varphi)/\varphi d\mu, n \in \mathbb{N}\}$  taking advantage of the gradient bounds on  $\varphi$  and  $d\mu/d\nu_\rho$  by an integration by parts formula.

## 4.1 Technical prerequisites

We first define a family of functionals,  $\{\Gamma_n, n \in \mathbb{N}\}$ , on  $\mathcal{D}_\rho^+ \times \mathcal{M}_\rho$ , whose limit when  $n$  tends to infinity is shown to exist.

**Lemma 4.1** *Assume  $d \geq 3$ . For  $\varphi \in \mathcal{D}_\rho^+$  and  $\mu \in \mathcal{M}_\rho$ , and any integer  $n$ , the functional  $\Gamma_n(\varphi, \mu) := \int \mathcal{L}_n(\varphi)/\varphi d\mu$  is well defined. If we call  $\tilde{\Gamma}_n(h, \mu) := \Gamma_n(\exp(h), \mu)$ , then for any  $\mu$ , the map  $h \mapsto \tilde{\Gamma}_n(h, \mu)$  is convex on the convex set  $\mathcal{E}_\rho$ .*

*Proof.* The formal full expression of  $\Gamma_n(\varphi, \mu)$  is

$$\Gamma_n(\varphi, \mu) = \sum_{i,j \in \Lambda_n} p(i, j) \int g_i \frac{\nabla_j^i \varphi}{\varphi} d\mu + \sum_{i \in \Lambda_n} \left( \gamma_\rho p_n^*(i, i) \int \frac{\nabla_i^+ \varphi}{\varphi} d\mu + p_n(i, i) \int g_i \frac{\nabla_i^- \varphi}{\varphi} d\mu \right). \quad (4.1)$$

Note that as  $\varphi \in \mathcal{D}_\rho^+$ ,  $T_j^i \varphi \leq A_i^- \varphi$ . Thus, (4.1) is defined if we bound  $\int g_i A_i^-(\varphi)/\varphi d\mu$  for each site  $i \in \Lambda_n$ . This is done as in (3.19).

From (4.1), an expression for  $\tilde{\Gamma}_n(h, \mu)$  is as follows

$$\begin{aligned} \tilde{\Gamma}_n(h, \mu) &= \sum_{i,j \in \Lambda_n} p(i, j) \int g_i \left( e^{\nabla_j^i h} - 1 \right) d\mu \\ &\quad + \sum_{i \in \Lambda_n} \left( \gamma_\rho p_n^*(i, i) \int (e^{\nabla_i^+ h} - 1) d\mu + p_n(i, i) \int g_i (e^{\nabla_i^- h} - 1) d\mu \right). \end{aligned} \quad (4.2)$$

The convexity of  $h \mapsto \tilde{\Gamma}_n(h, \mu)$  follows from the convexity of the exponential. ■

We now express  $\tilde{\Gamma}_n(h, \mu)$  in terms of gradients of  $h$  and  $\mu$ .

**Lemma 4.2** *For  $h \in \mathcal{E}_\rho$  and  $\mu \in \mathcal{M}_\rho$ , we have with  $f := d\mu/d\nu_\rho$*

$$\tilde{\Gamma}_n(h, \mu) = \sum_{i,j \in \Lambda_n} \gamma_\rho p_n(i, j) \left( \int (e^{\Delta_i^j h} - 1) \nabla_i^+ f d\nu_\rho + \int (e^{\Delta_i^j h} - 1 - \Delta_i^j h) d\mu \right) + R_n(h, \mu), \quad (4.3)$$

with,

$$\lim_{n \rightarrow \infty} \sup_{h \in \mathcal{E}_\rho} \sup_{\mu \in \mathcal{M}_\rho} |R_n(h, \mu)| = 0. \quad (4.4)$$

Note also that

$$\int \mathcal{L}_n(\varphi) d\mu = \sum_{i,j \in \Lambda_n} \gamma_\rho p_n(i, j) \int \Delta_i^j \varphi \nabla_i^+ f d\nu_\rho - \sum_{i \in \Lambda_n} \gamma_\rho p_n(i, i) \int \nabla_i^+ \varphi \nabla_i^+ f d\nu_\rho. \quad (4.5)$$

*Proof.* First, we apply the integration by parts formula (2.9) to (4.2):

$$\tilde{\Gamma}_n(h, \mu) = \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} p(i, j) \int g_i A_i^- \left( e^{\Delta_i^j h} - 1 \right) d\mu$$

$$\begin{aligned}
& + \sum_{i \in \Lambda_n} \left( p_n^*(i, i) \gamma_\rho \int (e^{\nabla_i^+ h} - 1) d\mu + p_n(i, i) \int g_i \mathbf{A}_i^-(e^{-\nabla_i^+ h} - 1) d\mu \right) \\
= & \gamma_\rho \sum_{i, j \in \Lambda_n} p_n(i, j) \left( \int (e^{\Delta_i^j h} - 1) \nabla_i^+ f d\nu_\rho + \int (e^{\Delta_i^j h} - 1 - \Delta_i^j h) d\mu \right) \\
& + R_n(h, \mu) + N(h, \mu), \tag{4.6}
\end{aligned}$$

with

$$\begin{aligned}
R_n(h, \mu) &:= \sum_{i \in \Lambda_n} \gamma_\rho p_n(i, i) \int (e^{-\nabla_i^+ h} - 1) \nabla_i^+(f) d\nu_\rho \\
&+ \gamma_\rho \sum_{i \in \Lambda_n} p_n^*(i, i) \int (e^{\nabla_i^+ h} - 1 - \nabla_i^+ h) d\mu \\
&+ \gamma_\rho \sum_{i \in \Lambda_n} p_n(i, i) \int (e^{-\nabla_i^+ h} - 1 + \nabla_i^+ h) d\mu, \tag{4.7}
\end{aligned}$$

and,

$$N(h, \mu) := \sum_{i, j \in \Lambda_n} \gamma_\rho p(i, j) \int \Delta_i^j h d\mu + \sum_{i \in \Lambda_n} \gamma_\rho (p_n^*(i, i) - p_n(i, i)) \int (\nabla_i^+ h) d\mu. \tag{4.8}$$

To show that  $N(h, \mu)$  vanishes, first write

$$\begin{aligned}
\sum_{i, j \in \Lambda_n} p(i, j) \int (\nabla_j^+ h) d\mu &= \sum_{j \in \Lambda_n} \left( \sum_{i \in \Lambda_n} p(i, j) \right) \int (\nabla_j^+ h) d\mu \\
&= \sum_{j \in \Lambda_n} \left( 1 - \sum_{i \notin \Lambda_n} p(i, j) \right) \int (\nabla_j^+ h) d\mu \\
&= \sum_{j \in \Lambda_n} (1 - p_n^*(j, j)) \int (\nabla_j^+ h) d\mu, \tag{4.9}
\end{aligned}$$

and similarly,

$$\sum_{i, j \in \Lambda_n} p_n(i, j) \int (\nabla_i^+ h) d\mu = \sum_{i \in \Lambda_n} (1 - p_n(i, i)) \int (\nabla_i^+ h) d\mu.$$

It is thus clear that  $N(h, \mu) = 0$ .

We now show that  $R_n(h, \mu)$  defined in (4.7) is negligible. Note that for  $i \notin \mathcal{S}$ ,  $\epsilon_i < 1$ . Also, for  $n$  large enough, if we define  $\partial^R \Lambda_n := \Lambda_n \setminus \Lambda_{n-R}$ , then  $\partial^R \Lambda_n \cap \mathcal{S} = \emptyset$ . Also, by (2.1) (iii),  $p_n(i, i) = 0$  when  $i \notin \partial^R \Lambda_n$ . Thus, there is a constant  $c_0 > 0$  such that for  $i \in \partial^R \Lambda_n$   $|\nabla_i^+ h| \leq -\log(1 - \epsilon_i) \leq c_0 \epsilon_i$ ,  $\nu_\rho$ -a.s., and  $|\nabla_i^+ f| \leq (\epsilon_i + \epsilon_i^*) f$ ,  $\nu_\rho$ -a.s. Thus, there is a constant  $c_1 > 0$  such that

$$p_n(i, i) \left| \int (e^{-\nabla_i^+ h} - 1) \nabla_i^+ f d\nu_\rho \right| \leq \int (\exp(c_0 \epsilon_i) - 1) (\epsilon_i + \epsilon_i^*) d\mu \leq c_1 \epsilon_i (\epsilon_i + \epsilon_i^*), \tag{4.10}$$

and by expanding to second order in  $\nabla_i^+ h$

$$|p_n^*(i, i)| \int (e^{\nabla_i^+ h} - 1 - \nabla_i^+ h) d\mu \leq c_1 \epsilon_i^2, \quad \text{and} \quad |p_n(i, i)| \int (e^{-\nabla_i^+ h} - 1 + \nabla_i^+ h) d\mu \leq c_1 \epsilon_i^2, \quad (4.11)$$

Combining (4.10) and (4.11), and summing over  $i \in \partial^R \Lambda_n$ , we obtain the desired asymptotics (4.4), since for dimension  $d \geq 3$ ,  $\sum \epsilon_i^2 < \infty$ .

We obtain (4.5) from (4.3) by setting  $h = \epsilon \varphi$  and expanding  $\tilde{\Gamma}_n(h, \mu)$  to first order in  $\epsilon$ . ■

We are now ready for the key technical lemma of this section.

**Proposition 4.3** *For  $(\varphi, \mu) \in \mathcal{D}_\rho^+ \times \mathcal{M}_\rho$ ,  $\{\Gamma_n(\varphi, \mu), n \in \mathbb{N}\}$  is a Cauchy sequence whose limit we denote by  $\Gamma_\infty(\varphi, \mu)$ . We have the following properties.*

(i) *For  $h \in \mathcal{E}_\rho$ ,  $h \mapsto \tilde{\Gamma}_\infty(h, \mu) := \Gamma_\infty(e^h, \mu)$  is convex.*

(ii) *The Cauchy sequence is uniform in the following sense*

$$\lim_{n \rightarrow \infty} \sup_{\varphi \in \mathcal{D}_\rho^+} \sup_{\mu \in \mathcal{M}_\rho} |\Gamma_n(\varphi, \mu) - \Gamma_\infty(\varphi, \mu)| = 0. \quad (4.12)$$

(iii) *For any integer  $n$ , and any  $\mu \in \mathcal{M}_\rho$  we denote by  $\mu_n$  the measure of  $\mathcal{M}_n$  of density  $f_n := E_{\nu_\rho}[d\mu/d\nu_\rho | \mathcal{F}_{\Lambda_n}]$ . Then,*

$$\lim_{n \rightarrow \infty} \sup_{\mu \in \mathcal{M}_\rho} \sup_{\varphi_n \in \mathcal{D}_n} |\Gamma_\infty(\varphi_n, \mu_n) - \Gamma_\infty(\varphi_n, \mu)| = 0. \quad (4.13)$$

(iv) *For  $\varphi_n \in \mathcal{D}_n^+$  and  $\mu_n \in \mathcal{M}_n$ , we have  $\Gamma_\infty(\varphi_n, \mu_n) = \Gamma_n(\varphi_n, \mu_n)$ .*

*Proof.* Step 1: We show that  $\{\Gamma_n(\varphi, \mu), n \in \mathbb{N}\}$  is a Cauchy sequence and (4.12) holds.

By using the expression (4.3) of Lemma 4.2, we have for  $m > n$

$$\begin{aligned} \tilde{\Gamma}_m(h, \mu) - \tilde{\Gamma}_n(h, \mu) &= \sum_{\substack{(i,j) \in \Lambda_m^2 \setminus \Lambda_n^2 \\ i \neq j}} \gamma_\rho p(i, j) \left( \int (e^{\Delta_i^j h} - 1) \nabla_i^+ f d\nu_\rho + \int (e^{\Delta_i^j h} - 1 - \Delta_i^j h) d\mu \right) \\ &\quad + R_m(h, \mu) - R_n(h, \mu). \end{aligned} \quad (4.14)$$

Since  $p(i, j) = 0$  when  $|i - j| > R$ , we can assume  $n$  and  $m$  so large that if  $(i, j) \in \Lambda_m^2 \setminus \Lambda_n^2$  with  $p(i, j) > 0$ , then  $i, j \notin \mathcal{S}$ . Thus, there is a positive constant  $c_0$  such that  $\nu_\rho$ -a.s.

$$\forall (i, j) \in \Lambda_m^2 \setminus \Lambda_n^2 \text{ with } p(i, j) > 0, \quad |\nabla_i^+ h| \leq -\log(1 - \epsilon_i) \leq c_0 \epsilon_i, \quad \text{and} \quad |\nabla_i^+ f| \leq (\epsilon_i + \epsilon_i^*) f. \quad (4.15)$$

Also, there is a positive constant  $c_1$  such that

$$\begin{aligned} |p(i, j)| \int (e^{\Delta_i^j h} - 1) \nabla_i^+ f d\nu_\rho &\leq \int (e^{c_0(\epsilon_i + \epsilon_j)} - 1) (\epsilon_i + \epsilon_i^*) f d\nu_\rho \\ &\leq c_1 (\epsilon_i + \epsilon_j) (\epsilon_i + \epsilon_i^*). \end{aligned} \quad (4.16)$$

Now, recalling that for  $i \notin \mathcal{S}$ ,  $\sum_j p(i, j)\epsilon_j = \epsilon_i$ , and  $\sum_j p(i, j) = 1$ , we have

$$\sum_{(i,j) \in \Lambda_m^2 \setminus \Lambda_n^2} c_1 p(i, j)(\epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_i^*) \leq 2c_1 \sum_{i \in \Lambda_n^c \cup \partial^R \Lambda_n} \epsilon_i(\epsilon_i + \epsilon_i^*) \xrightarrow{n \rightarrow \infty} 0, \quad (4.17)$$

since  $\sum_i \epsilon_i^2 = \sum_i (\epsilon_i^*)^2 < \infty$  when  $d \geq 3$ . Similarly, the second integral in (4.14) will go to 0, after we perform a second order expansion and use (4.15). Now, from Lemma 4.2,  $|R_m(h, \mu) - R_n(h, \mu)|$  converges to 0 uniformly in  $\mathcal{E}_\rho$  and  $\mathcal{M}_\rho$ .

Step 2: The limit  $h \mapsto \tilde{\Gamma}_\infty(h, \mu)$  is convex, since it is a pointwise limit of convex functions.

Step 3: We prove (4.13).

Let  $\varphi_n$  be in  $\mathcal{D}_n$ , and set  $h_n = \log(\varphi_n)$ . Note that for  $i \notin \Lambda_n$ ,  $\nabla_i^+ h_n = 0$ . Also, for any function  $\psi$ ,  $\mathbf{A}_i^+ E_{\nu_\rho}[\psi | \mathcal{F}_{\Lambda_n}] = E_{\nu_\rho}[\mathbf{A}_i^+ \psi | \mathcal{F}_{\Lambda_n}]$ . Thus, for  $m > R + n$

$$\begin{aligned} \tilde{\Gamma}_m(h_n, \mu) &= \tilde{\Gamma}_n(h_n, \mu) + \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \Lambda_n}} \gamma_\rho p(i, j) \int \left[ (e^{\nabla_j^+ h_n} - 1) \nabla_i^+ f + (e^{\nabla_j^+ h_n} - \nabla_j^+ h_n - 1) f \right] d\nu_\rho \\ &\quad + \sum_{\substack{j \in \Lambda_m \setminus \Lambda_n \\ i \in \Lambda_n}} \gamma_\rho p(i, j) \int \left[ (e^{-\nabla_i^+ h_n} - 1) \nabla_i^+ f + (e^{-\nabla_i^+ h_n} + \nabla_i^+ h_n - 1) f \right] d\nu_\rho \\ &\quad + R_m(h_n, \mu) - R_n(h_n, \mu). \end{aligned} \quad (4.18)$$

By observing that  $\tilde{\Gamma}_n(h_n, \mu) = \tilde{\Gamma}_n(h_n, \mu_n)$ , and that  $R_m(h_n, \mu) = 0$  for  $m > n + R$ , we have

$$\begin{aligned} \tilde{\Gamma}_m(h_n, \mu) - \tilde{\Gamma}_n(h_n, \mu_n) &= \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \partial^R \Lambda_n}} \gamma_\rho p(i, j) \int (e^{\nabla_j^+ h_n} - 1) \nabla_i^+ f d\nu_\rho \\ &\quad \sum_{j \in \partial^R \Lambda_n} \gamma_\rho p_n(j, j) \left( \int (e^{-\nabla_j^+ h_n} - 1) \nabla_j^+ (f) d\nu_\rho + \int (e^{-\nabla_j^+ h_n} + \nabla_j^+ h_n - 1) f d\nu_\rho \right) \\ &\quad \sum_{j \in \partial^R \Lambda_n} \gamma_\rho p_n^*(j, j) \int (e^{\nabla_j^+ h_n} - \nabla_j^+ h_n - 1) f d\nu_\rho - R_n(h_n, \mu). \end{aligned} \quad (4.19)$$

Now, using again that for  $j \in \partial^R \Lambda_n$ ,  $|\nabla_j^+ h_n| \leq c_0 \epsilon_j$ , and for  $i \in \partial^R \Lambda_n \cup \Lambda_n^c$ ,  $|\nabla_i^+ f| \leq (\epsilon_i + \epsilon_i^*) f$ , we have a constant  $C_1$  such that

$$\begin{aligned} |\tilde{\Gamma}_m(h_n, \mu) - \tilde{\Gamma}_n(h_n, \mu_n)| &= |R_n(h_n, \mu)| + C_1 \sum_{\substack{i \in \Lambda_m \setminus \Lambda_n \\ j \in \Lambda_n}} p(i, j) \epsilon_j (\epsilon_i + \epsilon_i^*) \\ &\quad + 2C_1 \sum_{j \in \partial^R \Lambda_n} \epsilon_j^2 + C_1 \sum_{j \in \partial^R \Lambda_n} \epsilon_j (\epsilon_j + \epsilon_j^*) \\ &\leq |R_n(h_n, \mu)| + 2C_1 \sum_{i \notin \Lambda_n} \epsilon_i^2 + (\epsilon_i^*)^2 + C_1 \sum_{j \in \partial^R \Lambda_n} (4\epsilon_j^2 + (\epsilon_j^*)^2) \end{aligned} \quad (4.20)$$

Equation (4.13) follows after we take the limit  $m$  to infinity in (4.20) and use (4.4) of Lemma 4.2.

Step 4: We show that  $\Gamma_\infty(\varphi_n, \mu_n) = \Gamma_n(\varphi_n, \mu_n)$ . Indeed, for  $m > R + n$ ,  $\mathcal{L}_{(m)}(\varphi_n) = \mathcal{L}(\varphi_n)$  so that

$$\Gamma_m(\varphi_n, \mu_n) = \int E_{\nu_\rho} \left[ \frac{\mathcal{L}_{(m)}(\varphi_n)}{\varphi_n} f_n \middle| \mathcal{F}_{\Lambda_m} \right] d\nu_\rho = \int \frac{E_{\nu_\rho}[\mathcal{L}_{(m)}(\varphi_n) | \mathcal{F}_{\Lambda_n}]}{\varphi_n} f_n d\nu_\rho = \Gamma_n(\varphi_n, \mu_n). \quad (4.21)$$

Now, a minimax theorem for  $\Gamma_\infty$  will be a corollary of Lemma 2.5.

**Proposition 4.4** *A minimax theorem holds for  $\Gamma_\infty$ . In other words,*

$$\sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho} \Gamma_\infty(\varphi, \mu) = \inf_{\varphi \in \mathcal{D}_\rho} \sup_{\mu \in \mathcal{M}_\rho} \Gamma_\infty(\varphi, \mu). \quad (4.22)$$

*Proof.* We need to check that for any  $\varphi \in \mathcal{D}_\rho$ , the map  $\mu \mapsto \Gamma_\infty(\varphi, \mu)$  on  $\mathcal{M}_\rho$  is continuous on the compact space  $\mathcal{M}_\rho$ . Let  $\{\mu_k, k \in \mathbb{N}\}$  be in  $\mathcal{M}_\rho$ , converging weakly to  $\mu \in \mathcal{M}_\rho$ . By Lemma 7.2, all densities  $f_k = d\mu_k/d\nu_\rho$  are uniformly bounded in  $L^2(\nu_\rho)$ . Thus  $f_k$  converges in weak- $L^2(\nu_\rho)$  to  $d\mu/d\nu_\rho$ . Now, for  $\varphi \in \mathcal{D}_\rho^+$ , as in (3.19),  $g_i(\varphi \circ T_j^i)/\varphi \in L^2(\nu_\rho)$ , so that for  $i, j \in \Lambda_n$

$$\int g_i \frac{\varphi \circ T_j^i}{\varphi} d\mu_k \xrightarrow{k \rightarrow \infty} \int g_i \frac{\varphi \circ T_j^i}{\varphi} d\mu.$$

Thus,  $\Gamma_n(\varphi, \mu_k) \rightarrow \Gamma_n(\varphi, \mu)$  as  $k \rightarrow \infty$ . Now, the uniform Cauchy property (4.12) implies that  $\Gamma_\infty(\varphi, \mu_k) \rightarrow \Gamma_\infty(\varphi, \mu)$  as  $k \rightarrow \infty$ .  $\blacksquare$

## 4.2 Proof of Theorem 1.3

If  $u_n$  is the principal normalized eigenfunction of  $\mathcal{L}_n$ , then for any  $n$  and any  $\mu_n$ , we have by Proposition 4.3 (iv)

$$\Gamma_\infty(u_n, \mu_n) = -\lambda_n(\rho). \quad (4.23)$$

Now, by (4.13) of Proposition 4.3, for any  $\epsilon > 0$ , there is  $n_0$  such that for any  $n \geq n_0$

$$\sup_{\mu \in \mathcal{M}_\rho} |\Gamma_\infty(u_n, \mu) - \Gamma_\infty(u_n, \mu_n)| \leq \epsilon. \quad (4.24)$$

Thus, for any  $\mu \in \mathcal{M}_\rho$  and  $n \geq n_0$

$$\Gamma_\infty(u_n, \mu) \leq -\lambda_n(\rho) + \epsilon \implies \inf_{\varphi \in \mathcal{D}_\rho} \Gamma_\infty(\varphi, \mu) \leq -\lambda_n(\rho) + \epsilon \quad (\text{since } \mathcal{D}_n \subset \mathcal{D}_\rho). \quad (4.25)$$

Recalling Lemma 1.1, and taking the limit  $n \rightarrow \infty$ , we obtain

$$\sup_{\mu \in \mathcal{M}_\rho} \inf_{\varphi \in \mathcal{D}_\rho} \Gamma_\infty(\varphi, \mu) \leq -\lambda(\rho). \quad (4.26)$$

Conversely, if  $h_n = E_{\nu_\rho}[h | \mathcal{F}_n]$  for  $h \in \mathcal{E}_\rho$ , we show by a convexity argument that

$$\forall \mu_n \quad \tilde{\Gamma}_\infty(h, \mu_n) \geq \tilde{\Gamma}_\infty(h_n, \mu_n) - \epsilon_n \quad \text{with} \quad \lim_{n \rightarrow \infty} \epsilon_n = 0. \quad (4.27)$$



Indeed, take  $m > n$  and in expression (4.3) break down the gradient  $\nabla_i^+ f_n$  so as to obtain

$$\tilde{\Gamma}_m(h, \mu_n) = \sum_{i,j \in \Lambda_m} \gamma_\rho p(i, j) \int \left[ (e^{\Delta_i^j h} - 1) \mathbf{A}_i^+ f_n - (\Delta_i^j h) f_n \right] d\nu_\rho + R_m(h, \mu_n). \quad (4.28)$$

We further divide the sum over  $\Lambda_m$  into

$$\tilde{\Gamma}_m(h, \mu_n) = \sum_{i,j \in \Lambda_n} \gamma_\rho p(i, j) \int \left[ (e^{\Delta_i^j h} - 1) \mathbf{A}_i^+ f_n - \Delta_i^j h f_n \right] d\nu_\rho + Q_{n,m}(h, \mu_n), \quad (4.29)$$

where  $Q_{n,m}(h, \mu_n)$  contains the sum over  $(i, j) \in \Lambda_m^2 \setminus \Lambda_n^2$ . With similar estimates as those showing that  $R_n(h, \mu)$  goes to 0 when  $n$  tends to infinity uniformly in  $h$  and  $\mu$ , in the proof of Proposition 4.3,  $Q_{n,m}(h, \mu)$  goes to 0 as  $n$  and  $m$  tend to infinity. Using that for any function  $\psi$  and  $i \in \Lambda_n$ ,  $\mathbf{A}_i^+ E_{\nu_\rho}[\psi | \mathcal{F}_n] = E_{\nu_\rho}[\mathbf{A}_i^+ \psi | \mathcal{F}_n]$ , we have by Jensen's inequality for the conditional expectation

$$\begin{aligned} \tilde{\Gamma}_m(h, \mu_n) &= \sum_{i,j \in \Lambda_n} \gamma_\rho p(i, j) \int E_{\nu_\rho}[e^{\Delta_i^j h} - 1 | \mathcal{F}_n] \mathbf{A}_i^+ f_n - E_{\nu_\rho}[\Delta_i^j h | \mathcal{F}_n] f_n d\nu_\rho + Q_{n,m}(h, \mu_n) \\ &\geq \sum_{i,j \in \Lambda_n} \gamma_\rho p(i, j) \int (e^{\Delta_i^j h_n} - 1) \mathbf{A}_i^+ f_n - \Delta_i^j(h_n) f_n d\nu_\rho + Q_{n,m}(h, \mu_n) \\ &\geq \tilde{\Gamma}_n(h_n, \mu_n) - Q_n(h_n, \mu_n) + Q_{n,m}(h, \mu_n) \\ &= \tilde{\Gamma}_\infty(h_n, \mu_n) - Q_n(h_n, \mu_n) + Q_{n,m}(h, \mu_n). \end{aligned} \quad (4.30)$$

Thus, by taking the limit as  $m$  tends to infinity, we obtain (4.27) with

$$\epsilon_n := \lim_{m \rightarrow \infty} \sup_{h, \mu} (|R_{n,m}(h, \mu)| + |Q_n(h_n, \mu_n)|) \xrightarrow{n \rightarrow \infty} 0. \quad (4.31)$$

Now, for any  $h \in \mathcal{E}_\rho$ , since  $\infty > \int \exp(h_n) u_n^* d\nu_\rho > 0$ , we can define

$$\frac{d\mu_n^*}{d\nu_\rho} = \frac{e^{h_n} u_n^*}{\int e^{h_n} u_n^* d\nu_\rho}.$$

Thus, by duality  $\tilde{\Gamma}_n(h_n, \mu_n^*) = \Gamma_n^*(u_n^*, \mu_n^*) = -\lambda_n(\rho)$ . and,

$$\sup_{\mu \in \mathcal{M}_\rho} \tilde{\Gamma}_\infty(h, \mu) \geq \tilde{\Gamma}_\infty(h, \mu_n^*) \geq \tilde{\Gamma}_\infty(h_n, \mu_n^*) - \epsilon_n = -\lambda_n(\rho) - \epsilon_n. \quad (4.32)$$

Thus, by taking the limit  $n$  to infinity, and using Lemma 1.1, we obtain

$$\sup_{\mu \in \mathcal{M}_\rho} \Gamma_\infty(e^h, \mu) \geq -\lambda(\rho) \implies \inf_{\varphi \in \mathcal{D}_\rho^+} \sup_{\mu \in \mathcal{M}_\rho} \Gamma_\infty(\varphi, \mu) \geq -\lambda(\rho). \quad (4.33)$$

Now, since by Proposition 4.4, the minimax Theorem holds for  $\Gamma_\infty$  the proof concludes.

## 5 Uniqueness: Proofs of Theorems 1.4 and 1.2.

The proofs of Theorem 1.4 and Theorem 1.2 will follow from three observations, which we have written as separate lemmas. First, any limit point of  $\{u_n\}$  solves (1.4(i)) and belongs to  $\mathcal{D}_\rho^+$ : this is shown in Lemmas 3.7 and 5.1. Second, solutions of (1.4(i)) in  $\mathcal{D}_\rho^+$  satisfy  $\Gamma_\infty(u, \mu) + \lambda(\rho) = 0$  for any  $\mu \in \mathcal{M}_\rho$ : this is shown in Lemma 5.2. Third, by convexity of  $h \mapsto \Gamma_\infty(\exp(h), \mu)$  shown in Proposition 4.3, there is a unique solution of  $\Gamma_\infty(u, \mu) + \lambda(\rho) = 0$  for any  $\mu \in \mathcal{M}_\rho$ : this is shown in Lemma 5.3.

**Lemma 5.1** *If  $u \in \mathcal{D}_\rho$ ,  $\int u d\nu_\rho = 1$ , and  $u$  satisfies (1.4(i)), then  $u$  is positive  $\nu_\rho$ -a.s. on  $\mathcal{A}^c$ .*

*Proof.* We denote by  $\mathcal{B} := \{\eta : u(\eta) = 0\}$ . Since  $u \in \mathcal{D}_\rho$ , we have for  $i \notin \mathcal{S}$  and  $\eta$   $\nu_\rho$ -a.s.,

$$u(\eta) \geq u(A_i^+ \eta) \quad \text{and} \quad u(A_i^+ \eta) \geq \frac{1}{1 - \epsilon_i} u(\eta).$$

Thus, for  $i \notin \mathcal{S}$ ,  $\mathcal{B} = (A_i^+)^{-1}(\mathcal{B})$   $\nu_\rho$ -a.s. . For any cylinder  $\theta$  with base in  $\mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , we will consider  $\mathcal{B}_\theta := \mathcal{B} \cap \theta$ . If  $T^{i,j}$  denotes the exchange operator at site  $i, j \in \mathbb{Z}^d$ , then

$$\mathcal{B}_\theta \stackrel{\nu_\rho\text{-a.s.}}{=} (A_i^+)^{-1}(\mathcal{B}_\theta), \quad \forall i \notin \mathcal{S} \implies \mathcal{B}_\theta \stackrel{\nu_\rho\text{-a.s.}}{=} (T^{i,j})^{-1}(\mathcal{B}_\theta), \quad \forall i, j \notin \mathcal{S}.$$

Indeed,

$$\mathcal{B}_\theta \stackrel{\nu_\rho\text{-a.s.}}{=} \bigcup_{k,l \in \mathbb{N}} \mathcal{B}_\theta \cap \{\eta(i) = k, \eta(j) = l\},$$

so that we can go from

$$\mathcal{B}_\theta \cap \{\eta(i) = k, \eta(j) = l\} \quad \text{to} \quad \mathcal{B}_\theta \cap \{\eta(i) = l, \eta(j) = k\} = (T^{i,j})^{-1}(\mathcal{B}_\theta \cap \{\eta(i) = k, \eta(j) = l\})$$

by a finite succession of creation and annihilation of particles. Now, by Hewitt-Savage 0-1 law on the lattice  $\mathbb{Z}^d \setminus \mathcal{S}$ , we conclude that  $\nu_\rho(\mathcal{B}_\theta) \in \{0, 1\}$ . Assume that for some cylinder  $\theta$ ,  $\nu_\rho(\mathcal{B}_\theta) = 1$ . Since  $u$  satisfies (1.4(i)) and  $1_\theta \in \mathbb{L}_b$ , we have

$$\int u \mathcal{L}^*(1_\theta) d\nu_\rho = 0 \implies \sum_{i,j \in \mathbb{Z}^d} p^*(i,j) \int g_i(\eta) u(\eta) 1_\theta(T_j^i \eta) d\nu_\rho = 0. \quad (5.1)$$

Now,

$$(T_j^i)^{-1}(\theta) = \begin{cases} T_i^j(\theta) & \text{if } \theta(j) > 0, \quad \text{and} \quad \emptyset \text{ if } \theta(j) = 0 \quad \text{when} \quad i, j \in \mathcal{S} \\ A_i^+(\theta) & \text{when} \quad i \in \mathcal{S}, j \notin \mathcal{S} \\ A_j^-(\theta) & \text{if } \theta(j) > 0, \quad \text{and} \quad \emptyset \text{ if } \theta(j) = 0 \quad \text{when} \quad i \notin \mathcal{S}, j \in \mathcal{S} \\ \theta & \text{when} \quad i, j \notin \mathcal{S} \end{cases} \quad (5.2)$$

Since the moves on the right hand side generates all cylinders with base in  $\mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , we obtain

$$\forall \theta \in \mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}, \quad \int_\theta u d\nu_\rho = 0, \quad (5.3)$$

which is absurd since  $\int u d\nu_\rho = 1$ . Thus,  $\nu_\rho(\mathcal{B}) = 0$  and the proof is concluded.  $\blacksquare$

**Lemma 5.2** *If  $u \in \mathcal{D}_\rho$  satisfies (1.4(i)) then  $\Gamma_\infty(u, \mu) = -\lambda(\rho)$ , for any  $\mu \in \mathcal{M}_\rho$ .*

*Proof.* Let  $u$  satisfies (1.4(i)). By Lemma 5.1,  $u \in \mathcal{D}_\rho^+$ . For any  $\varphi_n \in \mathbb{L}_b$  with  $\varphi_n$   $\mathcal{F}_n$ -measurable, we write (1.4(i)) as

$$\int \mathcal{L}^*(\varphi_n) u d\nu_\rho + \lambda(\rho) \int \varphi_n u d\nu_\rho = 0. \quad (5.4)$$

We make the standard integration by parts and use cancellations as in (4.8) to obtain

$$\begin{aligned} \int \mathcal{L}^*(\varphi_n) u d\nu_\rho &= \sum_{i,j \in \Lambda_n} \gamma_\rho p^*(i,j) \int \Delta_i^j \varphi_n \mathbf{A}_i^+ u d\nu_\rho - \sum_{i \in \Lambda_n} \sum_{j \notin \Lambda_n} \gamma_\rho p^*(i,j) \int \nabla_i^+ \varphi_n \mathbf{A}_i^+ u d\nu_\rho \\ &\quad + \sum_{i \notin \Lambda_n} \sum_{j \in \Lambda_n} \gamma_\rho p^*(i,j) \int \nabla_j^+ \varphi_n \mathbf{A}_i^+ u d\nu_\rho \\ &= \sum_{i,j \in \Lambda_n} \gamma_\rho p^*(i,j) \int \Delta_i^j(\varphi_n) \nabla_i^+(u) d\nu_\rho + \tilde{R}_n(\varphi_n), \end{aligned} \quad (5.5)$$

where

$$\tilde{R}_n(\varphi_n) = - \sum_{i \in \Lambda_n} \gamma_\rho p_n^*(i,i) \int \nabla_i^+ \varphi_n \nabla_i^+ u d\nu_\rho + \sum_{i \notin \Lambda_n} \sum_{j \in \Lambda_n} \gamma_\rho p_n(j,i) \int \nabla_j^+ \varphi_n \nabla_i^+ u d\nu_\rho. \quad (5.6)$$

Now, for any  $\mu \in \mathcal{M}_\rho$  with density  $f$ , it is easy to note that for a fix large integer  $M$ ,

$$\varphi_n^{(M)} := E_{\nu_\rho} \left[ \frac{f}{u} \wedge M | \mathcal{F}_n \right] \in \mathbb{L}_b,$$

and if we set  $\varphi = f/u$  and  $\varphi^{(M)} = (f/u) \wedge M$ , then both  $\varphi^{(M)}$  and  $\varphi$  are in  $L^p(\nu_\rho)$  for any integer  $p$ , and are such that for  $i$  large enough  $|\nabla_i^+(\psi)| \leq 2\psi(\epsilon_i + \epsilon_i^*)$ . Indeed, for  $i \notin \mathcal{S}$

$$u \geq \mathbf{A}_i^+ u \geq u(1 - \epsilon_i), \quad \text{and} \quad f \geq \mathbf{A}_i^+ f \geq f(1 - \epsilon_i - \epsilon_i^*). \quad (5.7)$$

Thus, if  $i$  is such that  $1 - \epsilon_i - \epsilon_i^* > 0$ ,

$$\frac{f}{u}(1 - \epsilon_i - \epsilon_i^* - 1) \leq \nabla_i^+ \left( \frac{f}{u} \right) \leq \frac{f}{u} \left( \frac{1}{1 - \epsilon_i} - 1 \right). \quad (5.8)$$

Thus, for  $i$  large enough  $|\nabla_i^+(\varphi)| \leq 2\varphi(\epsilon_i + \epsilon_i^*)$ . Also, since  $f \in L^p(\nu_\rho)$  and  $1_{\mathcal{A}^c}/u \in L^p(\nu_\rho)$  for any integer  $p$  by Lemma 7.2, we obtain that  $\varphi \in L^p(\nu_\rho)$  for any  $p$ . The same is true for  $\varphi^{(M)}$  after a simple algebra.

By a reasoning by now standard, since  $\varphi_n^{(M)}$  satisfies a bound like (5.8)

$$|\tilde{R}_n(\varphi_n^{(M)})| \leq c_1 \sum_{i \in \partial^R \Lambda_n} (\epsilon_i^2 + (\epsilon_i^*)^2) \int \varphi_n^{(M)} u d\nu_\rho \leq c_1 \|\varphi\|_{\nu_\rho} \|u\|_{\nu_\rho} \sum_{i \in \partial^R \Lambda_n} \epsilon_i^2 + (\epsilon_i^*)^2 \xrightarrow{n \rightarrow \infty} 0. \quad (5.9)$$

Recall that for  $i \in \Lambda_n$ ,  $\mathbf{A}_i^+ \varphi_n^{(M)} = E_{\nu_\rho}[\mathbf{A}_i^+ \varphi^{(M)} | \mathcal{F}_{\Lambda_n}]$ . Now, since  $\varphi^{(M)} \in L^2(\nu_\rho)$ , and  $\{\varphi_n^{(M)}, n \in \mathbb{N}\}$  is a positive martingale, we have that  $\{\varphi_n^{(M)}\}$  converges to  $\varphi^{(M)}$  in  $L^2(\nu_\rho)$  and a.s. . Thus, for any  $\psi \in L^2(\nu_\rho)$ , and  $i, j \in \Lambda_n$

$$\lim_{n \rightarrow \infty} \int \varphi_n^{(M)} \psi d\nu_\rho = \int \varphi^{(M)} \psi d\nu_\rho, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int \nabla_j^+ \varphi_n^{(M)} \nabla_i^+ \psi d\nu_\rho = \int \nabla_j^+ \varphi^{(M)} \nabla_i^+ \psi d\nu_\rho. \quad (5.10)$$

Thus, combining (5.5), (5.9) and (5.10) we obtain (the series being absolutely convergent)

$$\sum_{i,j \in \mathbb{Z}^d} \gamma_\rho p^*(i,j) \int \Delta_i^j \left( \frac{f}{u} \wedge M \right) \nabla_i^+ u d\nu_\rho + \lambda(\rho) \int \left( \frac{f}{u} \wedge M \right) u d\nu_\rho = 0. \quad (5.11)$$

An identical expression to (5.11) is also valid for  $f/u$  as we take the limit  $M$  to infinity.

We will now show that  $\Gamma_\infty(u, \mu)$  has the same expression as the first term of (5.11). Now, by taking the limit  $n$  to infinity in expression (4.5), we obtain

$$\lim_{n \rightarrow \infty} \int \mathcal{L}_n(u) \frac{f}{u} d\nu_\rho = \sum_{i,j \in \mathbb{Z}^d} \gamma_\rho p^*(i,j) \int \Delta_i^j \frac{f}{u} \nabla_i^+ u d\nu_\rho. \quad (5.12)$$

Indeed, (4.5) only requires that  $f/u \in L^p(\nu_\rho)$  and that for  $i$  large enough  $|\nabla_i^+(\frac{f}{u})| \leq 2\frac{f}{u}(\epsilon_i + \epsilon_i^*)$ . Finally, since  $\Gamma_\infty(u, \mu) = \lim_{n \rightarrow \infty} \Gamma_n(u, \mu)$ , (5.11) concludes the proof.  $\blacksquare$

**Lemma 5.3** *If  $u, \tilde{u} \in \mathcal{D}_\rho^+$ , and for any  $\mu \in \mathcal{M}_\rho$   $\Gamma_\infty(u, \mu) = \Gamma_\infty(\tilde{u}, \mu) = -\lambda(\rho)$ , and  $\int u d\nu_\rho = \int \tilde{u} d\nu_\rho$ , then  $u = \tilde{u}$   $\nu_\rho$ -a.s. .*

*Proof.* We can define

$$h := \log(u), \quad \text{and} \quad \tilde{h} := \log(\tilde{u}), \quad \text{with} \quad h, \tilde{h} \in \mathcal{E}_\rho.$$

Now, for  $\gamma \in ]0, 1[$ , we form  $h_\gamma = \gamma h + (1 - \gamma)\tilde{h}$ , and by convexity of  $\tilde{\Gamma}_n$ , for any  $\mu \in \mathcal{M}_\rho$ ,

$$0 \leq a_n(\mu) := \gamma \tilde{\Gamma}_n(h, \mu) + (1 - \gamma) \tilde{\Gamma}_n(\tilde{h}, \mu) - \tilde{\Gamma}_n(h_\gamma, \mu) \xrightarrow{n \rightarrow \infty} -\lambda(\rho) - \tilde{\Gamma}_\infty(h_\gamma, \mu), \quad (5.13)$$

where we used Lemma 5.2. Now, Lemma 5.2 is also valid for any  $u^*$  limit point of  $u_n^*$ , the principal eigenfunction of  $\mathcal{L}_n^*$ . Note that since  $u, \tilde{u} \in \mathcal{D}_\rho^+$  then  $u, \tilde{u} \in L^2(\nu_\rho)$ . By Jensen, this implies that  $\exp(h_\gamma) \in L^2(\nu_\rho)$  and  $\int \exp(h_\gamma) u^* d\nu_\rho < \infty$ . Finally, Lemma 5.1 would imply that  $u^*|_{\mathcal{A}^c} > 0$   $\nu_\rho$ -a.s. , so that  $\int u^* \exp(h_\gamma) d\nu_\rho > 0$ , and we can define

$$d\mu^* := \frac{e^{h_\gamma} u^* d\nu_\rho}{\int e^{h_\gamma} u^* d\nu_\rho} \in \mathcal{M}_\rho. \quad (5.14)$$

Now, by duality, and Lemma 5.2 applied to  $\mathcal{L}^*$ .

$$\tilde{\Gamma}_n(h_\gamma, \mu^*) = \int \frac{\mathcal{L}_n^*(u^*)}{u^*} d\mu^* = \Gamma_n^*(u^*, \mu^*) \xrightarrow{n \rightarrow \infty} -\lambda(\rho) = \tilde{\Gamma}_\infty(h_\gamma, \mu^*). \quad (5.15)$$

Thus,  $a_n(\mu^*)$  vanishes as  $n$  tends to  $\infty$ . However, for any  $i, j \in \mathbb{Z}^d$  and  $n$  large enough

$$a_n(\mu^*) \geq p(i, j) \int g_i A_{i,j} d\mu^*, \quad \text{with} \quad 0 \leq A_{i,j} := \gamma e^{\nabla_j^i h} + (1 - \gamma) e^{\nabla_j^i \tilde{h}} - e^{(\gamma \nabla_j^i h + (1 - \gamma) \nabla_j^i \tilde{h})}. \quad (5.16)$$

Now  $a_n(\mu^*) \rightarrow 0$ ,  $e^{h_\gamma} u^* > 0$   $\nu_\rho$ -a.s. on  $\mathcal{A}^c$ , and (5.16) imply that  $p(i, j) g_i A_{i,j} = 0$   $\nu_\rho$ -a.s. on  $\mathcal{A}^c$ . This in turn, implies that for  $\eta(i) p(i, j) > 0$ ,  $\nu_\rho$ -a.s., we have  $\nabla_j^i h = \nabla_j^i \tilde{h}$  in  $\mathcal{A}^c$ . Let us denote  $f := \tilde{u}/u$  on  $\mathcal{A}^c$ . Since,  $p(\cdot, \cdot)$  is irreducible, we obtain

$$\forall i, j \quad \text{with} \quad \eta(i) p(i, j) > 0 \quad f(T_j^i \eta) = f(\eta), \quad \nu_\rho - \text{a.s.} \quad (5.17)$$

This in turn, implies that for  $i, j \notin \mathcal{S}$   $f(T^{i,j}\eta) = f(\eta)$   $\nu_\rho$ -a.s. , so that by Hewitt-Savage 0-1 law for exchangeable events, we conclude that  $f$  is  $\nu_\rho$ -a.s. constant on each cylinder  $\theta$  with base in  $\mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , say  $c_\theta := f|_\theta$ .

We now show that the constants  $\{c_\theta\}$  are the same. Assume  $\theta, \theta' \in \mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$  with  $T_j^i \theta = \theta'$ . If we denote  $X_\theta := f^{-1}(\{c_\theta\})$ , then

$$\theta \subset X_\theta, \quad \theta' = T_j^i \theta \subset (T_i^j)^{-1}(X_\theta), \quad \text{and by (5.17)} \quad (T_i^j)^{-1}(X_\theta) \stackrel{\nu_\rho^{\text{a.s.}}}{=} X_\theta. \quad (5.18)$$

This yields  $c_\theta = c_{\theta'}$ . Assume now that for  $j \in \mathcal{S}$  with

$$\sum_{i \notin \mathcal{S}} p(i, j) > 0, \quad \text{we have} \quad \theta' = A_j^+ \theta.$$

Take  $i \notin \mathcal{S}$  with  $p(i, j) > 0$ , and note that

$$\theta' = A_j^+ \theta \subset (T_i^j)^{-1}(X_\theta) \stackrel{\nu_\rho^{\text{a.s.}}}{=} X_\theta \quad [\text{by (5.17)}] \quad (5.19)$$

Thus,  $c_\theta = c_{\theta'}$  in this case also. Now, we have assumed that  $\mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$  was a connected set containing  $0_{\mathcal{S}}$ . Thus, by a succession of moves  $T_i^j$  and  $A_j^+$  applied to  $0_{\mathcal{S}}$ , we cover all of  $\mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , and conclude that  $f$  is constant  $\nu_\rho$ -a.s. .  $\blacksquare$

## 6 Hitting time: Proof of Theorem 1.5.

Let  $u$  (resp.  $u^*$ ) be the principal Dirichlet eigenfunction of  $\mathcal{L}$  (resp.  $\mathcal{L}^*$ ) in  $\mathcal{D}_\rho$  (resp.  $\mathcal{D}_\rho^*$ ). By Lemma 5.1,  $u$  and  $u^*$  are  $\nu_\rho$ -a.s. positive on  $\mathcal{A}^c$ . Thus, we define a Markov semi-group on  $\mathcal{A}^c$ ,

$$\forall \eta \in \mathcal{A}^c, \quad S_t^u(\varphi)(\eta) := e^{\lambda(\rho)t} \frac{\bar{S}_t(u\varphi)}{u(\eta)}. \quad (6.1)$$

This semi-group is stationary with respect to

$$d\hat{\mu}_\rho = \frac{uu^*d\nu_\rho}{\int uu^*d\nu_\rho}. \quad (6.2)$$

Note that since  $1_{\mathcal{A}^c}/u \in L^2(\hat{\mu}_\rho)$  by Lemma 7.4, we have by definition, for all  $\eta \in \mathcal{A}^c$ ,

$$S_t^u\left(\frac{1_{\mathcal{A}^c}}{u}\right)(\eta) = c_t \frac{u_t(\eta)}{u(\eta)} \quad \text{with} \quad c_t = e^{\lambda(\rho)t} P_{\nu_\rho}(\tau > t), \quad \text{and} \quad u_t := \frac{P_\eta(\tau > t)}{P_{\nu_\rho}(\tau > t)}. \quad (6.3)$$

From (1.13),  $c_t \in [\underline{c}, 1]$ , whereas  $u_t \in \mathcal{D}_\rho$  from inequality (4.7) of [2]. It is then easy to check directly that, for any  $t > 0$

$$u S_t^u\left(\frac{1_{\mathcal{A}^c}}{u}\right) \in \mathcal{D}_\rho \implies \text{if } \psi_t := \frac{1}{t} \int_0^t S_s^u\left(\frac{1_{\mathcal{A}^c}}{u}\right) ds, \quad \text{then } u\psi_t \in \mathcal{D}_\rho. \quad (6.4)$$

Now, by Jensen's inequality,  $\{S_t^u, t > 0\}$  is a contraction semi-group on  $L^2(\mathcal{A}^c, \hat{\mu}_\rho)$ . Thus, by von Neumann's mean ergodic theorem in Hilbert space (see e.g. [12] Th.1.2 page 24), we obtain

$$\psi_t \xrightarrow{t \rightarrow \infty} \psi \quad \text{in } L^2(\hat{\mu}_\rho), \quad \text{and for any } t \geq 0 \quad S_t^u(\psi) = \psi, \quad \hat{\mu}_\rho - \text{a.s.} \quad (6.5)$$

If  $\{\psi_t\}$  converge to  $\psi$  in  $L^2(\hat{\mu}_\rho)$ , then  $\{\psi_t u\}$  converge weakly towards  $\psi u$ . Indeed, for any  $\varphi$  bounded and continuous

$$\begin{aligned} \left| \int \varphi u (\psi_t - \psi) d\nu_\rho \right| &\leq \left( \int u u^* d\nu_\rho \right) \left\| \frac{\varphi}{u^*} \right\|_{\hat{\mu}_\rho} \|\psi_t - \psi\|_{\hat{\mu}_\rho} \\ &\leq |\varphi|_\infty \left( \int u u^* d\nu_\rho \int \frac{u}{u^*} d\nu_\rho \right)^{1/2} \|\psi_t - \psi\|_{\hat{\mu}_\rho}. \end{aligned} \quad (6.6)$$

Since  $\int \psi_t u d\nu_\rho = \int_0^t c_s ds / t \leq 1$ , the Remark 2.6 yields that  $u\psi \in \mathcal{D}_\rho$ . Finally, since  $\hat{\mu}_\rho$  and  $\nu_\rho$  are equivalent in  $\mathcal{A}^c$ , (6.5) implies that for any  $t \geq 0$ ,

$$\bar{S}_t(\psi u) = e^{-\lambda(\rho)t} \psi u \text{ on } \mathcal{A}^c \text{ } \nu_\rho - \text{a.s.} \quad (6.7)$$

Thus, by differentiating (6.7) at  $t = 0$ , we obtain that  $\psi u$  is a Dirichlet principal eigenfunction in  $\mathcal{D}_\rho^+$ , with  $\underline{c} \leq \int u \psi d\nu_\rho \leq 1$ . By Theorem 1.4, this means that  $\psi$  is constant. To find the value of  $\psi$ , integrate (6.5) against  $1_{\mathcal{A}^c}$ .

$$\psi \equiv \int \psi d\hat{\mu}_\rho = \lim_{t \rightarrow \infty} \int \frac{1}{t} \int_0^t S_s^u \left( \frac{1}{u} \right) ds d\hat{\mu}_\rho = \int \frac{1}{u} d\hat{\mu}_\rho = \frac{1}{\int u u^* d\nu_\rho}. \quad (6.8)$$

Finally, since  $1/u^* \in L^2(\hat{\mu}_\rho)$ , we integrate (6.5) against  $1/u^*$  to conclude the proof with

$$\frac{1}{t} \int_0^t \frac{e^{\lambda(\rho)s} P_{\nu_\rho}(\tau > s)}{\int u u^* d\nu_\rho} ds = \int \frac{1}{u^*} \psi_t d\hat{\mu}_\rho \xrightarrow{t \rightarrow \infty} \int \frac{1}{u^*} \psi d\hat{\mu}_\rho = \frac{1}{(\int u u^* d\nu_\rho)^2}. \quad (6.9)$$

■

## 7 Appendix

We have often used Lemma 7.2 below to obtain regularity of probability densities satisfying a gradient bound (1.6) [3, 4, 2, 5]. For ease of reading, we recall its simple proof. Then, in Lemma 7.4, we show how similar arguments yield the regularity of  $1_{\mathcal{A}^c}/\varphi$  for  $\varphi \in \mathcal{D}_\rho^+$ .

For ease of writing, we identify a cylinder with its base. Thus, when we write  $\theta \in \mathbb{N}^{\mathcal{S}}$ , we mean  $\theta := \{\eta \in \mathbb{N}^{\mathbb{Z}^d} : \eta(i) = \theta(i), \forall i \in \mathcal{S}\}$ . Recalling the notations used in the definition of  $\nu_\rho$  (see (2.3)), let  $\nu_\epsilon$  be the product measure

$$d\nu_\epsilon(\eta) = \prod_{i \in \mathcal{S}} d\theta_{\gamma(\rho)}(\eta_i) \prod_{i \notin \mathcal{S}} d\theta_{(1-\epsilon_i)\gamma(\rho)}(\eta_i).$$

We showed in [2] that when  $d \geq 3$ ,  $\nu_\epsilon$  is absolutely continuous with respect to  $\nu_\rho$ , and that if  $\psi_\epsilon := d\nu_\epsilon/d\nu_\rho$ , then for any integer  $p$

$$\int \psi_\epsilon^p d\nu_\rho < \infty, \quad \text{and} \quad \int \frac{1}{\psi_\epsilon^p} d\nu_\rho < \infty. \quad (7.1)$$

**Remark 7.1** The purpose of introducing  $\psi_\epsilon$  was that for any  $i \notin \mathcal{S}$ ,  $A_i^+ \psi_\epsilon = (1 - \epsilon_i) \psi_\epsilon$ . Thus, if  $\varphi \in \mathcal{D}_\rho$ , then  $\varphi/\psi_\epsilon$  is increasing outside  $\mathcal{S}$ . Indeed, using (2.11)(ii),

$$\forall i \notin \mathcal{S}, \quad A_i^+(\varphi/\psi_\epsilon) \geq \varphi/\psi_\epsilon.$$

**Lemma 7.2** We assume that  $d \geq 3$ . For any integer  $n$ , any  $\theta \in \mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , and  $\varphi \in \mathcal{D}_\rho$

$$\int_\theta \varphi^n d\nu_\rho \leq \left( \frac{\int_\theta \varphi d\nu_\rho}{\nu_\epsilon(\theta)} \right)^n \int_\theta \psi_\epsilon^n d\nu_\rho. \quad (7.2)$$

Also,

$$\int \varphi^n d\nu_\rho \leq C_n \left( \int \varphi d\nu_\rho \right)^n \quad \text{with} \quad C_n := \frac{\int \psi_\epsilon^n d\nu_\rho}{\nu_\epsilon(0_{\mathcal{S}})^{n+1}} < \infty. \quad (7.3)$$

*Proof.* We define the measure  $d\mu = \varphi d\nu_\rho$ , and for  $\theta \in \mathbb{N}^{\mathcal{S}} \setminus \mathcal{A}$ , we define two probability measures  $d\mu_\theta = 1_\theta d\mu / \mu(\theta)$  and  $d\nu_\theta = 1_\theta d\nu_\epsilon / \nu_\epsilon(\theta)$ . Note that on  $\theta$ , the probability measure  $\nu_\epsilon$  satisfies Holley's condition (see Theorem 2.9, p.75 in [10]) which implies that it satisfies FKG's inequality.

**Step 1.** We first show that for any  $\phi$  decreasing on  $\theta$ ,

$$\int \phi d\mu_\theta \leq \int \phi d\nu_\theta. \quad (7.4)$$

By the Remark 7.1,  $d\mu_\theta/d\nu_\theta$  is increasing in  $\theta$ . We apply FKG's inequality on  $\theta$

$$\int \phi d\mu_\theta = \int \phi \frac{d\mu_\theta}{d\nu_\theta} d\nu_\theta \leq \int \phi d\nu_\theta. \quad (7.5)$$

**Step 2.** First, note that  $\varphi$  and  $\psi_\epsilon = d\nu_\epsilon/d\nu_\rho$  are non-negative decreasing on  $\theta$ . So is  $\varphi^i \psi_\epsilon^j$  for any integers  $i, j$ . We apply (7.4) to  $\phi := \varphi^i \psi_\epsilon^j$  and obtain

$$\int_\theta \varphi^{i+1} \psi_\epsilon^j \frac{d\nu_\rho}{\mu(\theta)} = \int \varphi^i \psi_\epsilon^j d\mu_\theta \leq \int \varphi^i \psi_\epsilon^j d\nu_\theta = \int_\theta \varphi^i \psi_\epsilon^{j+1} \frac{d\nu_\rho}{\nu_\epsilon(\theta)}. \quad (7.6)$$

By induction, we obtain (7.2) for any integer  $n$ . Now, (7.3) obtains after taking  $\theta = 0_{\mathcal{S}}$  and using FKG's inequality once more. Indeed, since  $\varphi$  and  $1_{0_{\mathcal{S}}}$  are both decreasing

$$\int \varphi^n d\nu_\rho \leq \frac{\int \varphi^n 1_{0_{\mathcal{S}}} d\nu_\rho}{\nu_\rho(0_{\mathcal{S}})}. \quad (7.7)$$

■

**Remark 7.3** Actually if  $\mu \in \mathcal{M}_\rho$ , then its density  $f := d\mu/d\nu_\rho$  satisfies an inequality like (7.3) but with  $U := \{i : (\epsilon_i + \epsilon_i^*) \geq 1\}$  replacing  $\mathcal{S}$  which was the domain where  $\epsilon_i = 1$ . Since  $U$  is bounded,  $\nu_\epsilon(U) > 0$ .

**Lemma 7.4** *We assume that  $d \geq 3$ . Let  $\varphi \in \mathcal{D}_\rho^+$  and  $\theta \in \mathbb{N}^S \setminus \mathcal{A}$ . Then, for any integer  $n$*

$$\int_\theta \varphi^n d\nu_\rho \int_\theta \frac{1}{\varphi^n} d\nu_\rho \leq \int_\theta \psi_\epsilon^n d\nu_\rho \int_\theta \frac{1}{\psi_\epsilon^n} d\nu_\rho. \quad (7.8)$$

Furthermore, for

$$c_{\varphi,n} := \sup_{\theta \in \mathbb{N}^S \setminus \mathcal{A}} \left\{ \nu_\rho(\theta) \left( \int_\theta \varphi \frac{d\nu_\rho}{\nu_\rho(\theta)} \right)^{-n} \right\} < \infty,$$

we have

$$\int_{\mathcal{A}^c} \frac{1}{\varphi^n} d\nu_\rho \leq c_{\varphi,n} \int \psi_\epsilon^n d\nu_\rho \int_{\mathcal{A}^c} \frac{1}{\psi_\epsilon^n} d\nu_\rho. \quad (7.9)$$

*Proof.* Recall that  $\varphi/\psi_\epsilon$  is increasing on  $\mathcal{A}^c$  whereas for any integer  $n$ ,  $\varphi^n$  is decreasing. Thus, for any cylinder  $\theta \in \mathbb{N}^S \setminus \mathcal{A}$ , if we denote  $d\tilde{\nu}_\rho = 1_\theta d\nu_\rho / \nu_\rho(\theta)$ , then, by FKG's inequality

$$\int_\theta \frac{1}{\psi_\epsilon^n} d\tilde{\nu}_\rho = \int_\theta \left( \frac{\varphi}{\psi_\epsilon} \right)^n \frac{1}{\varphi^n} d\tilde{\nu}_\rho \geq \int_\theta \left( \frac{\varphi}{\psi_\epsilon} \right)^n d\tilde{\nu}_\rho \int_\theta \frac{1}{\varphi^n} d\tilde{\nu}_\rho. \quad (7.10)$$

Also, since  $\psi_\epsilon^n$  is decreasing

$$\int_\theta \left( \frac{\varphi}{\psi_\epsilon} \right)^n d\tilde{\nu}_\rho \int_\theta \psi_\epsilon^n d\tilde{\nu}_\rho \geq \int_\theta \left( \frac{\varphi}{\psi_\epsilon} \right)^n \psi_\epsilon^n d\tilde{\nu}_\rho = \int_\theta \varphi^n d\tilde{\nu}_\rho \quad (7.11)$$

Multiplying (7.10) by  $\int_\theta \psi_\epsilon^n d\tilde{\nu}_\rho$ , using (7.11), and simplifying by  $\int_\theta (\varphi/\psi_\epsilon)^n d\tilde{\nu}_\rho > 0$ , (since  $\varphi > 0$   $\nu_\rho$ -a.s.) we obtain (7.8). Note that  $c_\varphi < \infty$  since there is a finite number of elements in  $\mathbb{N}^S \setminus \mathcal{A}$ , on each of which  $\int_\theta \varphi d\nu_\rho > 0$ .

Finally, (7.9) is obtained by summing over all  $\theta \in \mathbb{N}^S \setminus \mathcal{A}$ , and applying Hölder's inequality to  $\int \varphi^n d\tilde{\nu}_\rho$ . ■

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